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A Note on Solutions of the Matrix Equation $\mathrm{AXB} = \mathrm{C}$

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Abstract: This paper deals with necessary and sufficient condition for consistency of the matrix equation $AXB = C$. We will be concerned with the minimal number of free parameters in Penrose's formula $X = A^{(1)}CB^{(1)} + Y - A^{(1)}AYBB^{(1)}$ for obtaining the general solution of the matrix equation and we will establish the relation between the minimal number of free parameters and the ranks of the matrices A and B. The solution is described in the terms of Rohde's general form of the $\{1\}$-inverse of the matrices A and B. We will also use Kronecker product to transform the matrix equation $AXB = C$ into the linear system $(B^T \otimes A)\vec{X} = \vec{C}$.

Keywords: Generalized inverses, Kronecker product, matrix equations, linear systems

1 Introduction

In this paper we consider matrix equation

$$AXB = C, \tag{1}$$

where X is an $n \times k$ matrix of unknowns, A is an $m \times n$ matrix of rank a, B is a $k \times l$ matrix of rank b, and C is an $m \times l$ matrix, all over $\mathbb{C}$. The set of all $m \times n$ matrices over the complex field $\mathbb{C}$ will be denoted by $\mathbb{C}^{m\times n}$, $m, n \in \mathbb{N}$. The set of all $m \times n$ matrices over the complex field $\mathbb{C}$ of rank a will be denoted by $\mathbb{C}^{m\times n}_a$. We will write $A_{i\rightarrow}$ ($A_{\downarrow j}$) for the i^{th} row (the j^{th} column) of the matrix $A \in \mathbb{C}^{m\times n}$ and $\vec{A}$ will denote an ordered stock of columns of A, i.e.

$$\vec{A} = \begin{bmatrix} A_{\downarrow 1} \\ A_{\downarrow 2} \\ \vdots \\ A_{\downarrow n} \end{bmatrix}.$$

Using the Kronecker product of the matrices B^T and A we can transform the matrix equation (1) into linear system

$$(B^T \otimes A)\vec{X} = \vec{C}. \tag{2}$$

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For the proof we refer the reader to A. Graham [2]. Necessary and sufficient condition for consistency of the linear system $Ax = c$, as well as the minimal number of free parameters in Penrose's formula $x = A^{(1)}c + (I - AA^{(1)})y$ has been considered in the paper B. Malešević, I. Jovović, M. Makragić and B. Radičić [3]. We will here briefly sketch this results in the case of the linear system (2).

Any matrix X satisfying the equality $AXA = A$ is called $\{1\}$-inverse of A and is denoted by $A^{(1)}$. For each matrix $A \in \mathbb{C}_a^{m\times n}$ there are regular matrices $P \in \mathbb{C}^{n\times n}$ and $Q \in \mathbb{C}^{m\times m}$ such that

$$QAP = E_A = \left[\begin{array}{c|c} I_a & 0 \\ \hline 0 & 0 \end{array}\right], \tag{3}$$

where I_a is $a \times a$ identity matrix. It can be easily seen that every $\{1\}$-inverse of the matrix A can be represented in the Rohde's form

$$A^{(1)} = P \left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array}\right] Q, \tag{4}$$

where $U = [u_{ij}]$, $V = [v_{ij}]$ and $W = [w_{ij}]$ are arbitrary matrices of corresponding dimensions $a \times (m-a)$, $(n-a) \times a$ and $(n-a) \times (m-a)$ with mutually independent entries, see C. Rohde [10] and V. Perić [8]. We will explore the minimal numbers of free parameters in Penrose's formula

$$X = A^{(1)}CB^{(1)} + Y - A^{(1)}AYBB^{(1)}$$

for obtaining the general solution of the matrix equation (1). Some similar considerations can be found in papers B. Malešević and B. Radičić [4], [5], [6] and [9].

2 Matrix equation *AXB=C* and the Kronecker product of the matrices B^T and A

The Kronecker product of matrices $A = [a_{ij}]_{m\times n} \in \mathbb{C}^{m\times n}$ and $B = [b_{ij}]_{k\times l} \in \mathbb{C}^{k\times l}$, denoted by $A \otimes B$, is defined as block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}.$$

The matrix $A \otimes B$ is $mk \times nl$ matrix with mn blocks $a_{ij}B$ of order $k \times l$. Here we will mention some properties and rules for the Kronecker product. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{k\times l}$, $C \in \mathbb{C}^{n\times r}$ and $D \in \mathbb{C}^{l\times s}$. Then the following propositions holds:

- $A^T \otimes B^T = (A \otimes B)^T$;
- $rank(A \otimes B) = rank(A)\ rank(B)$;

- $(A\otimes B)\cdot(C\otimes D)=(A\cdot C)\otimes(B\cdot D)$ (mixed product rule);
- if A and B are regular $n\times n$ and $k\times k$ matrices, then $(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}$.

The proof of these facts can be found in A. Graham [2] and A. Ben–Israel and T.N.E. Greville [1].
Matrix $A^{(1)}\otimes B^{(1)}$ is $\{1\}$-inverse of $A\otimes B$. Using mixed product rule we have

$$(A\otimes B)(A^{(1)}\otimes B^{(1)})(A\otimes B)=(A\cdot A^{(1)}\cdot A)\otimes(B\cdot B^{(1)}\cdot B)=A\otimes B\,.$$

Let $R\in\mathbb{C}^{k\times k}$ and $S\in\mathbb{C}^{l\times l}$ be regular matrices such that

$$RBS=E_B=\left[\begin{array}{c|c} I_b & 0\\ \hline 0 & 0\end{array}\right]. \tag{5}$$

An $\{1\}$-inverse of the matrix B can be represented in the Rohde's form

$$B^{(1)}=S\left[\begin{array}{c|c} I_b & M\\ \hline N & K\end{array}\right]R\,, \tag{6}$$

where $M=[m_{ij}]$, $N=[n_{ij}]$ and $K=[k_{ij}]$ are arbitrary matrices of corresponding dimensions $b\times(k-b)$, $(l-b)\times b$ and $(l-b)\times(k-b)$ with mutually independent entries.

From now on, we will look more closely at the linear system (2).
Firstly, by mixed product rule we obtain

$$(S^T\otimes Q)\cdot(B^T\otimes A)\cdot(R^T\otimes P)=(S^T\cdot B^T\cdot R^T)\otimes(Q\cdot A\cdot P)=E_{B^T}\otimes E_A\,.$$

Unfortunately, the matrix

$$E_{B^T}\otimes E_A=\left[\begin{array}{c|c} I_b\otimes E_A & 0\\ \hline 0 & 0\end{array}\right]=\left[\begin{array}{cccc|c} E_A & & & & \\ & E_A & & 0 & \\ & & \ddots & & 0\\ & 0 & & E_A & \\ \hline & & 0 & & 0\end{array}\right]=\left[\begin{array}{cccccccc|c} I_a & 0 & & & & & & & \\ 0 & 0 & & & & & 0 & & \\ & & I_a & 0 & & & & & \\ & & 0 & 0 & & & & & 0\\ & & & & \ddots & & & & \\ & 0 & & & & & I_a & 0 & \\ & & & & & & 0 & 0 & \\ \hline & & & & 0 & & & & 0\end{array}\right]$$

is not of the needed form $E_{B^T\otimes A}$. Equality $E_{B^T}\otimes E_A=E_{B^T\otimes A}$ holds for $b=1$. Swapping the rows and the columns corresponding to blocks I_a and to zero diagonal blocks we get required matrix $E_{B^T\otimes A}$. If matrices D and G are the elementary matrices obtained by swapping rows and columns corresponding to mentioned blocks of the identity matrices, then $D\cdot(E_{B^T}\otimes E_A)\cdot G=E_{B^T\otimes A}$. Thus, we have

$$(D\cdot(S^T\otimes Q))\cdot(B^T\otimes A)\cdot((R^T\otimes P)\cdot G)=E_{B^T\otimes A} \tag{7}$$

and so an $\{1\}$-inverse of the matrix $B^T\otimes A$ can be represented in the Rohde's form

$$(B^T\otimes A)^{(1)}=(R^T\otimes P)\cdot G\cdot\left[\begin{array}{c|c} I_{ab} & F\\ \hline H & L\end{array}\right]\cdot D\cdot(S^T\otimes Q)\,, \tag{8}$$

where $F=[f_{ij}]$, $H=[h_{ij}]$ and $L=[l_{ij}]$ are arbitrary matrices of corresponding dimensions $ab\times(ml-ab)$, $(nk-ab)\times ab$ and $(nk-ab)\times(ml-ab)$ with mutually independent entries. If the matrices A and B are square matrices, then $D=G^T$.

For the simplicity of notation, we will write $\overline{c}_a$ ($\underline{c}_a$) for the submatrix corresponding to the first (the last) a coordinates of the vector c. Now, we can rephrase Lemma 2.1. and Theorem 2.2. from the paper B. Malešević, I. Jovović, M. Makragić and B. Radičić [3] in the case of the linear system (2). Let C' be given by $C'=QCS$. Then $\vec{C}'=(S^T\otimes Q)\,\vec{C}$.

Lemma 2.1 *The linear system* (2) *has a solution if and only if the last $ml-ab$ coordinates of the vector $c''=D\vec{C}'$ are zeros, where D is elementary matrix such that* (7) *holds.*

Theorem 2.2 *The vector*

$$\vec{X}=(B^T\otimes A)^{(1)}\vec{C}+(I-(B^T\otimes A)^{(1)}\cdot(B^T\otimes A))y, \tag{9}$$

$y\in\mathbb{C}^{nk\times 1}$ is an arbitrary column, is the general solution of the system (2), *if and only if the $\{1\}$-inverse $(B^T\otimes A)^{(1)}$ of the system matrix $B^T\otimes A$ has the form* (8) *for arbitrary matrices F and L and the rows of the matrix $H(\overline{c}''_{ab}-\overline{y}'_{ab})+\underline{y}'_{nk-ab}$ are free parameters, where $D\cdot(S^T\otimes Q)\vec{C}=c''=\left[\frac{\overline{c}''_{ab}}{0}\right]$ and $G^{-1}\cdot((R^{-1})^T\otimes P^{-1})y=y'=\left[\frac{\overline{y}'_{ab}}{\underline{y}'_{nk-ab}}\right]$.*

In the paper B. Malešević, I. Jovović, M. Makragić and B. Radičić [3] we have seen that general solution (9) can be presented in the form

$$\vec{X}=(R^T\otimes P)\cdot G\cdot\left[\frac{\overline{c}''_{ab}}{H(\overline{c}''_{ab}-\overline{y}'_{ab})+\underline{y}'_{nk-ab}}\right]. \tag{10}$$

We illustrate this formula in the next example.

Example 2.3 *We consider the matrix equation*

$$AXB=C,$$

where $A=\begin{bmatrix}1&2\\0&1\\1&1\end{bmatrix}$, $X=\begin{bmatrix}x_{11}&x_{12}&x_{13}\\x_{21}&x_{22}&x_{23}\end{bmatrix}$, $B=\begin{bmatrix}1&0&0\\0&1&1\\1&1&1\end{bmatrix}$ *and* $C=\begin{bmatrix}-3&-6&-6\\-1&-2&-2\\-2&-4&-4\end{bmatrix}$.

Using the Kronecker product the matrix equation may be considered in the form of the equivalent linear system

$$(B^T\otimes A)\cdot\vec{X}=\vec{C},$$

i.e.

$$\begin{bmatrix}1&2&0&0&1&2\\0&1&0&0&0&1\\1&1&0&0&1&1\\0&0&1&2&1&2\\0&0&0&1&0&1\\0&0&1&1&1&1\\0&0&1&2&1&2\\0&0&0&1&0&1\\0&0&1&1&1&1\end{bmatrix}\cdot\begin{bmatrix}x_{11}\\x_{21}\\x_{12}\\x_{22}\\x_{13}\\x_{23}\end{bmatrix}=\begin{bmatrix}-3\\-1\\-2\\-6\\-2\\-4\\-6\\-2\\-4\end{bmatrix}.$$

Matrices $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ *and* $P = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ *are regular matrices such that* $QAP = E_A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and matrices* $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ *and* $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ *are regular matrices such that* $RBS = E_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. *Therefore,*

$$E_{B^T} \otimes E_A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and for matrix } D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

we have

$$E_{B^T \otimes A} = D \cdot (E_{B^T} \otimes E_A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let us remark that $E_{B^T \otimes A} = (D \cdot (S^T \otimes Q)) \cdot (B^T \otimes A) \cdot (R^T \otimes P)$, *and hence according to the Theorem 2.2* $\vec{X} = (R^T \otimes P) \cdot \begin{bmatrix} \overline{c}''_4 \\ \hline H(\overline{c}''_4 - \overline{y}'_4) + \underline{y}'_2 \end{bmatrix}$ *is the general solution of the linear system iff elements of the column* $H(\overline{c}''_4 - \overline{y}'_4) + \underline{y}'_2$ *are two mutually independent parameters*

α_1 i α_2 for the vector

$$c'' = \left[\begin{array}{c} \vec{c}_4'' \\ \hline 0 \end{array}\right] = (D\cdot(S^T\otimes Q))\cdot\vec{C} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -1 \\ -2 \\ -6 \\ -2 \\ -4 \\ -6 \\ -2 \\ -4 \end{bmatrix} = \left[\begin{array}{c} -3 \\ -1 \\ -6 \\ -2 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right].$$

Finally, the general solution of the linear system is

$$\vec{X} = (R^T\otimes P)\cdot \left[\begin{array}{c} -3 \\ -1 \\ -6 \\ -2 \\ \hline \alpha_1 \\ \alpha_2 \end{array}\right] = \begin{bmatrix} 1 & -2 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \left[\begin{array}{c} -3 \\ -1 \\ -6 \\ -2 \\ \hline \alpha_1 \\ \alpha_2 \end{array}\right] = \begin{bmatrix} -1-\alpha_1+2\alpha_2 \\ -1-\alpha_2 \\ -2-\alpha_1+2\alpha_2 \\ \alpha_2-2\alpha_3-\alpha_4 \\ -2-\alpha_2 \\ \alpha_1+2\alpha_2 \\ \alpha_2 \end{bmatrix}.$$

3 Matrix equation $AXB=C$ and the $\{1\}-$inverses of the matrices A and B

In this section we indicate how technique of an $\{1\}$-inverse may be used to obtain the necessary and sufficient condition for an existence of a general solution of the matrix equation (1) without using Kronecker product. We will use the symbols $^{\ulcorner}C_{a,b}$, $_{\llcorner}C_{a,b}$, $C^{\urcorner}{}_{a,b}$ and $C_{\lrcorner a,b}$ for the submatrices of the matrix C corresponding to the first a rows and b columns, the last a rows and the first b columns, the first a rows and the last b columns, the last a rows and b columns, respectively.

Lemma 3.1 *The matrix equation* (1) *has a solution if and only if the last $m-a$ rows and $l-b$ columns of the matrix $C'=QCS$ are zeros, where $Q\in\mathbb{C}^{m\times m}$ and $S\in\mathbb{C}^{l\times l}$ are regular matrices such that* (3) *and* (5) *hold.*

Proof: The matrix equation (1) has a solution if and only if $C=AA^{(1)}CB^{(1)}B$, see R. Penrose [7]. Since $A^{(1)}$ and $B^{(1)}$ are described by the equations (4) and (6), it follows that

$$AA^{(1)} = AP\left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array}\right]Q = Q^{-1}\left[\begin{array}{c|c} I_a & U \\ \hline 0 & 0 \end{array}\right]Q$$

and

$$B^{(1)}B = S\left[\begin{array}{c|c} I_b & M \\ \hline N & K \end{array}\right] RB = S\left[\begin{array}{c|c} I_b & 0 \\ \hline N & 0 \end{array}\right] S^{-1}.$$

Hence, since Q and S are regular matrices we have the following equivalences

$$C = AA^{(1)}CB^{(1)}B \iff \quad QCS = QAA^{(1)}CB^{(1)}BS \overset{C'=QCS}{\iff} C' = \left[\begin{array}{c|c} I_a & U \\ \hline 0 & 0 \end{array}\right] C' \left[\begin{array}{c|c} I_b & 0 \\ \hline N & 0 \end{array}\right]$$

$$\iff \quad \left[\begin{array}{c|c} {}^{\ulcorner}C'_{a,b} & C'^{\urcorner}_{a,l-b} \\ \hline {}_{\llcorner}C'_{m-a,b} & C'_{\lrcorner m-a,l-b} \end{array}\right] = \left[\begin{array}{c|c} I_a & U \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} {}^{\ulcorner}C'_{a,b} & C'^{\urcorner}_{a,l-b} \\ \hline {}_{\llcorner}C'_{m-a,b} & C'_{\lrcorner m-a,l-b} \end{array}\right] \left[\begin{array}{c|c} I_b & 0 \\ \hline N & 0 \end{array}\right]$$

$$\iff \quad \left[\begin{array}{c|c} {}^{\ulcorner}C'_{a,b} & C'^{\urcorner}_{a,l-b} \\ \hline {}_{\llcorner}C'_{m-a,b} & C'_{\lrcorner m-a,l-b} \end{array}\right] = \left[\begin{array}{c|c} {}^{\ulcorner}C'_{a,b} + U\,{}_{\llcorner}C'_{m-a,b} + C'^{\urcorner}_{a,l-b}N + U\,C'_{\lrcorner n-a,l-b}N & 0 \\ \hline 0 & 0 \end{array}\right],$$

for $C' = \left[\begin{array}{c|c} {}^{\ulcorner}C'_{a,b} & C'^{\urcorner}_{a,l-b} \\ \hline {}_{\llcorner}C'_{m-a,b} & C'_{\lrcorner m-a,l-b} \end{array}\right]$. Therefore, we conclude

$$C = AA^{(1)}CB^{(1)}B \iff C'^{\urcorner}_{a,l-b} = 0 \wedge {}_{\llcorner}C'_{m-a,b} = 0 \wedge C'_{\lrcorner m-a,l-b} = 0. \quad \square$$

As we have seen in the Lemma 2.1 the matrix equation (1) has a solution if and only if the last $ml-ab$ coordinates of the column $c'' = D\vec{C}'$ are zeros, where D is elementary matrix such that (7) holds. Here we obtain the same result without using Kronecker product. The last $m(l-b)$ elements of the column $\vec{C}'$ are zeros and there are b blocks of $m-a$ zeros. Multiplying by the left column $\vec{C}'$ with elementary matrix D switches the rows corresponding to this zeros blocks under the blocks ${}^{\ulcorner}C'_{a,b\downarrow i}$, $1 \le i \le b$. Hence, the last $m(l-b)+(m-a)b = ml-ab$ entries of the column c'' are zeros.

Furthermore, we give a new form of the general solution of the matrix equation (1) using $\{1\}$-inverses of the matrices A and B.

Theorem 3.2 *The matrix*

$$X = A^{(1)}CB^{(1)} + Y - A^{(1)}AYBB^{(1)}, \tag{11}$$

$Y \in \mathbb{C}^{n\times k}$ is an arbitrary matrix, is the general solution of the matrix equation (1) *if and only if the $\{1\}$-inverses $A^{(1)}$ and $B^{(1)}$ of the matrices A and B have the forms* (4) *and* (6) *for arbitrary matrices U, W, N and K and the entries of the matrices*

$$V\big(\underset{a,b}{{}^{\ulcorner}C'} - \underset{a,b}{{}^{\ulcorner}Y'}\big) + \underset{n-a,b}{{}_{\llcorner}Y'}, \quad \big(\underset{a,b}{{}^{\ulcorner}C'} - \underset{a,b}{{}^{\ulcorner}Y'}\big)M + \underset{a,k-b}{Y'^{\urcorner}}, \quad V\big(\underset{a,b}{{}^{\ulcorner}C'} - \underset{a,b}{{}^{\ulcorner}Y'}\big)M + \underset{n-a,k-b}{Y'_{\lrcorner}} \tag{12}$$

are mutually independent free parameters, where

$$QCS = C' = \left[\begin{array}{c|c} {}^{\ulcorner}C'_{a,b} & 0 \\ \hline 0 & 0 \end{array}\right] \quad \textit{and} \quad P^{-1}YR^{-1} = Y = \left[\begin{array}{c|c} {}^{\ulcorner}Y'_{a,b} & Y'^{\urcorner}_{a,k-b} \\ \hline {}_{\llcorner}Y'_{n-a,b} & Y'_{\lrcorner n-a,k-b} \end{array}\right]. \tag{13}$$

Proof: Since the $\{1\}$-inverses $A^{(1)}$ and $B^{(1)}$ of the matrices A and B have the forms (4) and (6), the solution of the system $X = A^{(1)}CB^{(1)} + Y - A^{(1)}AYBB^{(1)}$ can be represented in the form

$$X \;=\; P\left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array}\right] QCS \left[\begin{array}{c|c} I_b & M \\ \hline N & K \end{array}\right] R \;+\; Y \;-\; P\left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array}\right] QAPP^{-1}YR^{-1}RBS \left[\begin{array}{c|c} I_b & M \\ \hline N & K \end{array}\right] R.$$

According to Lemma 3.1 and from (3) and (5) we have

$$\begin{aligned} X \;=\; & P\left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array}\right] \left[\begin{array}{c|c} \ulcorner C'_{a,b} & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} I_b & M \\ \hline N & K \end{array}\right] R \;+\; Y \\ & -\; P\left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array}\right] \left[\begin{array}{c|c} I_a & 0 \\ \hline 0 & 0 \end{array}\right] P^{-1}YR^{-1} \left[\begin{array}{c|c} I_b & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} I_b & M \\ \hline N & K \end{array}\right] R. \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned} X \;&=\; P\left[\begin{array}{c|c} \ulcorner C'_{a,b} & \ulcorner C'_{a,b}M \\ \hline V\ulcorner C'_{a,b} & V\ulcorner C'_{a,b}M \end{array}\right] R + Y - P\left[\begin{array}{c|c} I_a & 0 \\ \hline V & 0 \end{array}\right] P^{-1}YR^{-1} \left[\begin{array}{c|c} I_b & M \\ \hline 0 & 0 \end{array}\right] R \\ &=\; P\left(\left[\begin{array}{c|c} \ulcorner C'_{a,b} & \ulcorner C'_{a,b}M \\ \hline V\ulcorner C'_{a,b} & V\ulcorner C'_{a,b}M \end{array}\right] + Y' - \left[\begin{array}{c|c} I_a & 0 \\ \hline V & 0 \end{array}\right] Y' \left[\begin{array}{c|c} I_b & M \\ \hline 0 & 0 \end{array}\right]\right) R \\ &=\; P\left(\left[\begin{array}{c|c} \ulcorner C'_{a,b} & \ulcorner C'_{a,b}M \\ \hline V\ulcorner C'_{a,b} & V\ulcorner C'_{a,b}M \end{array}\right] + \right. \\ &+\; \left. \left[\begin{array}{c|c} \ulcorner Y'_{a,b} & Y'^{\urcorner}{}_{a,k-b} \\ \hline \llcorner Y'_{n-a,b} & Y'_{\lrcorner n-a,k-b} \end{array}\right] - \left[\begin{array}{c|c} I_a & 0 \\ \hline V & 0 \end{array}\right] \left[\begin{array}{c|c} \ulcorner Y'_{a,b} & Y'^{\urcorner}{}_{a,k-b} \\ \hline \llcorner Y'_{n-a,b} & Y'_{\lrcorner n-a,k-b} \end{array}\right] \left[\begin{array}{c|c} I_b & M \\ \hline 0 & 0 \end{array}\right]\right) R \\ &=\; P\left(\left[\begin{array}{c|c} \ulcorner C'_{a,b} & \ulcorner C'_{a,b}M \\ \hline V\ulcorner C'_{a,b} & V\ulcorner C'_{a,b}M \end{array}\right] + \left[\begin{array}{c|c} \ulcorner Y'_{a,b} & Y'^{\urcorner}{}_{a,k-b} \\ \hline \llcorner Y'_{n-a,b} & Y'_{\lrcorner n-a,k-b} \end{array}\right] - \left[\begin{array}{c|c} \ulcorner Y'_{a,b} & \ulcorner Y'_{a,b}M \\ \hline V\ulcorner Y'_{a,b} & V\ulcorner Y'_{a,b}M \end{array}\right]\right) R, \end{aligned}$$

where $Y' = P^{-1}YR^{-1}$. We now conclude

$$X \;=\; P\left[\begin{array}{c|c} \ulcorner C'_{a,b} & (\ulcorner C'_{a,b} - \ulcorner Y'_{a,b})M + Y'^{\urcorner}{}_{a,k-b} \\ \hline V(\ulcorner C'_{a,b} - \ulcorner Y'_{a,b}) + \llcorner Y'_{n-a,b} & V(\ulcorner C'_{a,b} - \ulcorner Y'_{a,b})M + Y'_{\lrcorner n-a,k-b} \end{array}\right] R. \qquad (14)$$

According to the Theorem 2.2 the general solution of the equation (1) has $nk - ab$ free parameters. Therefore, since the matrices P and R are regular we deduce that the solution (14) is the general if and only if the entries of the matrices separately

$$V\big(\ulcorner C'_{a,b} - \ulcorner Y'_{a,b}\big) + \llcorner Y'_{n-a,b}\,, \;\; \big(\ulcorner C'_{a,b} - \ulcorner Y'_{a,b}\big)M + Y'^{\urcorner}{}_{a,k-b}\,, \;\; V\big(\ulcorner C'_{a,b} - \ulcorner Y'_{a,b}\big)M + Y'_{\lrcorner n-a,k-b}$$

are $nk - ab$ free parameters. □

We can illustrate the Theorem 3.2 on the following two examples.

Example 3.3 *Consider again the matrix equation from previous example*

$$A \times B = C,$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and } C = \begin{bmatrix} -3 & -6 & -6 \\ -1 & -2 & -2 \\ -2 & -4 & -4 \end{bmatrix}.$$

We have

$$C' = QCS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & -6 & -6 \\ -1 & -2 & -2 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -6 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\ulcorner C'_{2,2} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}.$$

The solution of the matrix equation $AXB = C$ is

$$X = P \left[\begin{array}{cc|c} -3 & -6 & \alpha_1 \\ -1 & -2 & \alpha_2 \end{array}\right] R = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \left[\begin{array}{cc|c} -3 & -6 & \alpha_1 \\ -1 & -2 & \alpha_2 \end{array}\right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1-\alpha_1+2\alpha_2 & -2-\alpha_1+2\alpha_2 & \alpha_1-2\alpha_2 \\ -1-\alpha_2 & -2-\alpha_2 & \alpha_2 \end{bmatrix}.$$

Example 3.4 *We now consider the matrix equation*

$$AXB = C,$$

where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$ *and* $C = \begin{bmatrix} 2 & -6 \\ 4 & -12 \\ 2 & -6 \end{bmatrix}$.

For regular matrices $Q = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ *and* $P = \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *the following equality* $QAP = E_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *holds. Thus, rank of the matrix A is $a = 1$. There* *are regular matrices* $R = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ *and* $S = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ *such that* $RBS = E_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *holds. Thus, rank of the matrix* B *is* $b = 1$. *Since the ranks of the matrices* A *and* B *are* $a = b = 1$, *according to the Lemma 3.1 all entries of the last column and two rows of the matrix* $C' = QCS$ *are zeros, i.e. we get that the matrix* C' *is of the form*

$$C' = QCS = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 \\ 4 & -12 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Applying the Theorem 3.2, we obtain the general solution of the given matrix equation

$$\begin{aligned} X = P \left[\begin{array}{c|c} 2 & \alpha_1 \\ \hline \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{array}\right] R &= \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|c} 2 & \alpha \\ \hline \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{array}\right] \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 - 2\alpha - 3\beta_1 - 2\beta_2 + 6\gamma_1 + 4\gamma_2 & \alpha - 3\gamma_1 - 2\gamma_2 \\ \beta_1 - 2\gamma_1 & \gamma_1 \\ \beta_2 - 2\gamma_2 & \gamma_2 \end{bmatrix}. \end{aligned}$$

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