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Remarks on Sequences Generated by Harmonic Numbers

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Abstract: A monotonicity of sequences generated by the harmonic numbers is proved in this paper.

Keywords: Harmonic numbers, monotony sequences, recurrent sequences.

1 Introduction

A sequence of harmonic numbers $(H_n), n \in N_0$, is defined as

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \qquad H_0 = 0.$$

This sequence plays an important role in solving numerous problems in combinatorial mathematics as well as during analysis of complexity of numerical algorithms (see for example [1, 2, 3, 4]).

Let $(x_n), n \in N_0$ be an arbitrary sequence of real numbers. An operator of k-th difference is defined as

$$\Delta^0 x_n = x_n, \qquad \Delta^1 x_n = x_{n+1} - x_n, \qquad \Delta^k x_n = \Delta(\Delta^{k-1} x_n),$$

for each $k \in N_0$ and $n \in N_0$. A real sequence $(x_n), n \in N_0$, with the property $\Delta^k x_n \geq 0$, ($\Delta^k x_n \leq 0$), is said to be k-convex (i.e. k-concave). In the special cases, for $k = 0$, a sequence (x_n) is non negative (non positive), when $k = 1$ it is monotone non decreasing (non increasing), and for $k = 2$ it is convex (concave).

In this paper we are going to determine the monotonicity of two sequences generated by the harmonic numbers.

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2 Main result

Lemma 1 *Let $H=(H_n), n\in N_0$, be a sequence of harmonic numbers. Then, for each $k, k\in N$, the following equality*

$$\Delta^k H_n=\frac{(-1)^{k-1}(k-1)!}{(n+1)(n+2)\cdots(n+k)} \tag{1}$$

is valid.

Proof. We will prove Lemma 1 by mathematical induction. Since

$$\Delta H_n=H_{n+1}-H_n=\frac{1}{n+1},$$

we have that equality (1) holds when $k=1$. Now, suppose that (1) is valid for some fixed $k:=k, k\in N$. In that case, for $k:=k+1$ we have

$$\begin{aligned}
\Delta^{k+1}H_n &= \Delta(\Delta^k H_n)=\Delta\left(\frac{(-1)^{k-1}(k-1)!}{(n+1)(n+2)\cdots(n+k)}\right)=\\
&= \frac{(-1)^{k-1}(k-1)!}{(n+2)(n+3)\cdots(n+k+1)}-\frac{(-1)^{k-1}(k-1)!}{(n+1)(n+2)\cdots(n+k)}=\\
&= \frac{(-1)^{k-1}(k-1)!}{(n+2)(n+3)\cdots(n+k)}\left(\frac{1}{n+k+1}-\frac{1}{n+1}\right)=\\
&= \frac{(-1)^k k!}{(n+1)(n+2)\cdots(n+k+1)}.
\end{aligned}$$

□

The important corollary of Lemma 1 is

Corollary 1 *For $k=2$ and $k=3$ the following two inequalities are valid*

$$\Delta^2 H_n=-\frac{1}{(n+1)(n+2)}<0 \tag{2}$$

and

$$\Delta^3 H_n=\frac{2}{(n+1)(n+2)(n+3)}>0 \tag{3}$$

for each $n\in N$.

Now, according to sequence $H=(H_n), n\in N_0$, we will define sequences $(a_n), n\in N_0$ and $(b_n), n\in N_0$, as

$$a_n=\frac{H_{n+1}-H_1}{n},\quad a_0\equiv 0,\qquad b_n=\frac{a_{n+1}-a_1}{n},\quad b_0\equiv 0, \tag{4}$$

for each $n\in N_0$. The following result is valid for the sequences (a_n) and (b_n), $n\in N_0$.

Theorem 1 *The sequences* (a_n) *and* (b_n), $n \in N_0$, *defined by (4), are monotone decreasing and increasing, respectively.*

Proof. From the inequality (2) and equality

$$\begin{aligned}\sum_{k=1}^{n} k\Delta^2 H_k &= \sum_{k=1}^{n} k \sum_{i=0}^{2} (-1)^i \binom{2}{i} H_{k+2-i} = \\ &= nH_{n+2} - (n+1)H_{n+1} + H_1 + \sum_{i=0}^{2} (-1)^i \binom{2}{i} \sum_{k=i+1}^{n-2+i} H_{k+2-i} = \\ &= n(H_{n+2} - H_1) - (n+1)(H_{n+1} - 1),\end{aligned}$$

we have that

$$\frac{H_{n+2} - H_1}{n+1} < \frac{H_{n+1} - H_1}{n},$$

so we conclude that the sequence (a_n), $n \in N_0$, id monotone decreasing.

Now we will prove that sequence $(b_n), n \in N_0$ is monotone increasing. We have that for each $n, n \geq 1$ the following equality is valid

$$\begin{aligned}\sum_{k=1}^{n} k\Delta^3 H_k &= \sum_{k=1}^{n} k \sum_{i=0}^{3} (-1)^i \binom{3}{i} H_{k+3-i} = \\ &= nH_{n+3} - (2n+1)H_{n+2} + (n+1)H_{n+1} + H_2 - H_1 = \\ &= n(n+2)\frac{H_{n+3} - H_1}{n+2} - (2n+1)(n+1)\frac{H_{n+2} - H_1}{n+1} + \\ &+ n(n+1)\frac{H_{n+1} - H_1}{n} + \frac{H_2 - H_1}{1} = \\ &= n(n+2)a_{n+2} - (2n+1)(n+1)a_{n+1} + n(n+1)a_n + a_1 = \\ &= n(n+2)(a_{n+2} - a_a) - (2n+1)(n+1)(a_{n+1} - a_1) + n(n+1)(a_n - a_1).\end{aligned}$$

From the above equality and inequality (3), we have that for each $k, k \geq 1$, holds the inequality

$$k(k+2)(a_{k+2} - a_1) - (2k+1)(k+1)(a_{k+1} - a_1) + k(k+1)(a_k - a_1) > 0.$$

By summing this inequality over k we have that

$$\begin{aligned}&\sum_{k=1}^{n} (k(k+2)(a_{k+1} - a_1) - (2k+1)(k+1)(a_{k+1} - a_1) + k(k+1)(a_k - a_1)) = \\ = \;&(n+2)(n(a_{n+2} - a_1) - (n+1)(a_{n+1} - a_1)) > 0,\end{aligned}$$

i.e.

$$b_{n+1} = \frac{a_{n+2} - a_1}{n+1} > \frac{a_{n+1} - a_1}{n} = b_n.$$

□.

3 Conclusion

We have defined two real sequences (a_n) and (b_n) based on the harmonic numbers. We have proved that sequence (a_n) is monotone decreasing, while (b_n) is monotone increasing.

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