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New 2D continuous symmetric Christoffel-Darboux formula for Chebyshev orthonormal polynomials of the first kind

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Abstract: In this paper, a new two-dimensional continuous symmetric Christoffel-Darboux formula for orthonormal classical Chebyshev polynomials of the first kind is proposed. This continuous two-dimensional function of two real variables is most directly applied to approximation problems and synthesis of filter functions. The examples of the proposed two-dimensional Christoffel-Darboux formula are illustrated.

Keywords: Christoffel-Darboux formula, Chebyshev polynomials of the first kind, two-dimensional functions, classical orthogonal functions

1 introduction

Application of the extremal properties of Christoffel-Darboux sum for orthonormal classical orthogonal polynomials is described in [1-6] for the synthesis of the continuous one-dimensional and two-dimensional digital filter functions.

Implementation of classical orthogonal polynomials in filter theory is given in [1-18]. In the paper [19] is described the application of finite real Christoffel-Darboux sum for continuous one-dimensional and continuous two-dimensional functions obtained by orthogonal Chebyshev polynomials of the second kind.

In this paper we are giving the continuation of the original study [19] for Chebyshev polynomials of the first kind. Paper [19] and this paper make a global entity and an excellent basis for the new functions in one-dimensional z domain, i.e. two-dimensional z domain. Explicit compact expression for the proposed functions is very useful for the efficient design of 1D and 2D continuous and digital filter functions, respectively. In the papers [20-26] the mathematical background for the classical orthogonal polynomials is given.

In the second section the polynomial functions of one real variable for Christoffel-Darboux formula for Chebyshev polynomials of the first kind are provided. In the third

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chapter the proposed new formula for 2D Christoffel-Darboux formula for Chebyshev orthogonal polynomials of the first kind is explained, and examples in 3D plot and 2D contour plot of the proposed 2D functions for two real variables of $n \times n$ order are illustrated.

In this paper, we propose an original two-dimensional symmetric Christoffel-Darboux formula for classical orthonormal Chebyshev polynomials of the first kind in the compact explicit form, which is valid for even and odd order. The examples are illustrated and tables of two-dimensional symmetric functions of low order are given.

2 One-dimensional classical Christoffel-Darbuova formula for ortonormal Chebyshev polynomials of the first kind

The general form of a polynomial function of n-th order is given in the following expression:

$$\phi_n(x) = \sum_{k=0}^{n} a_k x^k \tag{1}$$

Chebyshev orthogonal polynomials of n-th order of the first kind, $T_n(x)$, the real variable x, are given in the following expression:

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{[n/2]} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2x)^{n-2k} \tag{2}$$

Chebyshev orthogonal polynomials are orthogonal on interval [-1, 1] with the weighting function $\omega(x)$

$$\omega(x) = \frac{1}{\sqrt{1-x^2}} \tag{3}$$

Where the norm is

$$h_r = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_r(x) T_r(x)\, dx = \begin{cases} \pi, & r = 0 \\ \frac{\pi}{2}, & r \neq 0 \end{cases} \tag{4}$$

Example 1:
For even *n, n=12,* (i.e. *n=2r*) Chebyshev polynomials of the first kind, $T_{12}(x)$, has the form

$$T_{12}(x) = 1 - 72x^2 + 840x^4 - 3584x^6 + 6912x^8 - 6144x^{10} + 2048x^{12}$$

and it is shown in Fig. 1.

Example 2:
For odd *n, n=15,* (i.e. *n=2r+1*) Chebyshev polynomials of the first kind, $T_{15}(x)$, has the form

$$T_{15}(x) = -15x + 560x^3 - 6048x^5 + 28800x^7 - 70400x^9 + 92160x^{11} - 61440x^{13} + 16384x^{15}$$

and it is shown in Fig. 2.

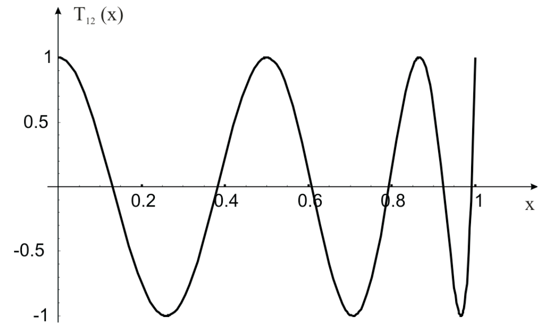

Fig. 1: Orthogonal Chebyshev polynomials of the first kind for even order n, n=12, in the normalized interval

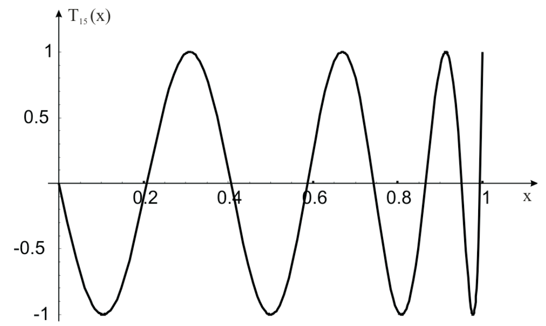

Fig. 2: Orthogonal Chebyshev polynomials of the first kind for odd order n, n=15, in the normalized interval

Extremal properties of the polynomial continuous classical one-dimensional real function is given by Christoffel-Darboux formula for Chebyshev orthogonal polynomials of the first kind and has the form:

$$\Phi_n(x) = \frac{[T_0(x)]^2}{h_0} + \frac{[T_1(x)]^2}{h_1} + \frac{[T_2(x)]^2}{h_2} + \ldots + \frac{[T_{n-1}(x)]^2}{h_{n-1}} + \frac{[T_n(x)]^2}{h_n} \tag{5}$$

Using formula 4, the expression 5 becomes as follows

$$\Phi_n(x) = \frac{2}{\pi}\left(\frac{1}{2}[T_0(x)]^2 + [T_1(x)]^2 + [T_2(x)]^2 + \ldots + [T_{n-1}(x)]^2 + [T_n(x)]^2\right) \tag{6}$$

i.e.

$$\frac{\pi}{2}\Phi_n(x) = \frac{1}{2}[T_0(x)]^2 + [T_1(x)]^2 + [T_2(x)]^2 + \ldots + [T_{n-1}(x)]^2 + [T_n(x)]^2 \tag{7}$$

or

$$\left(\frac{\pi}{2}\right)\Phi_n(x) = \frac{1}{2}T_0(x) + \sum_{r=1}^{n} T_r(x)\cdot T_r(x) \tag{8}$$

Based on the expression (4), the following table shows examples of polynomial functions of Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the first kind for different values of the polynomial order, n, $n \in (0, 1, 2, \ldots, 12)$.
From the equality (4) were performed fundamental observations [20], known in the literature as Bessel's and Parserval's formula.

Table 1: ONE-DIMENSIONAL CLASSICAL CHRISTOFFEL-DARBOUX FORMULA FOR GIVEN ORDER, n

n	$\left(\frac{\pi}{2}\right)\Phi_n(x) - \Phi_n(0) = \sum_{r=1}^{n} T_r(x)\cdot T_r(x)$
0	-2
1	$-2 + x^2$
2	$-1 - 3x^2 + 4x^4$
3	$-1 + 6x^2 - 20x^4 + 16x^6$
4	$-10x^2 + 60x^4 - 112x^6 + 64x^8$
5	$15x^2 - 140x^4 + 448x^6 - 576x^8 + 256x^{10}$
6	$1 - 21x^2 + 280x^4 - 1344x^6 + 2880x^8 - 2816x^{10} + 1024x^{12}$
7	$1 + 28x^2 - 504x^4 + 3360x^6 - 10560x^8 + 16896x^{10} - 13312x^{12} + 4096x^{14}$
8	$2 - 36x^2 + 840x^4 - 7392x^6 + 31680x^8 - 73216x^{10} + 93184x^{12} - 61440x^{14} + 16384x^{16}$
9	$2 + 45x^2 - 1320x^4 + 14784x^6 - 82368x^8 + 256256x^{10} - 465920x^{12} + 491520x^{14} - 278528x^{16} + 65536x^{18}$
10	$3 - 55x^2 + 1980x^4 - 27456x^6 + 192192x^8 - 768768x^{10} + 1863680x^{12} - 2785280x^{14} + 2506752x^{16} - 1245184x^{18} + 262144x^{20}$
11	$3 + 66x^2 - 2860x^4 + 48048x^6 - 411840x^8 + 2050048x^{10} - 6336512x^{12} + 12533760x^{14} - 15876096x^{16} + 12451840x^{18} - 5505024x^{20} + 1048576x^{22}$
12	$4 - 78x^2 + 4004x^4 - 80080x^6 + 823680x^8 - 4978688x^{10} + 19009536x^{12} - 47628288x^{14} + 79380480x^{16} - 87162880x^{18} + 60555264x^{20} - 24117248x^{22} + 4194304x^{24}$

For even order, $n = 12$, the Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the first kind $(\pi/2)\,[\Phi_{12}(x) - \Phi_0(x)/\pi]$is illustrated in Fig. 3.

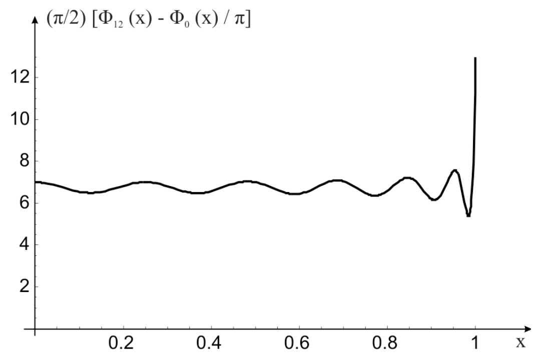

Fig. 3: Example of one-dimensional classical Christoffel-Darboux sum, $(\pi/2)\,\Phi_{12}(x)$, for orthonormal Chebyshev polynomials of the first kind for even order n, n=12, in the normalized interval

For odd order,$n = 15$, normalized Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the first kind $[\phi_{15}(x) - \Phi_0(0)]/[\Phi_{15}(1) - \Phi_0(0)$ is illustrated in Fig. 4.

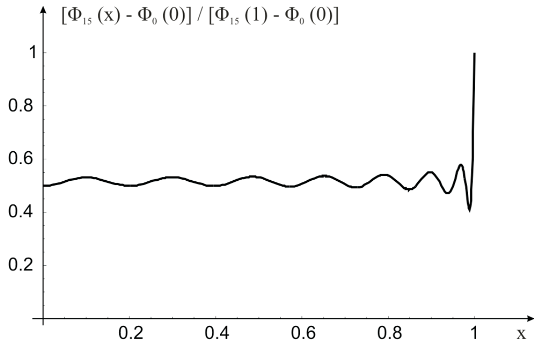

Fig. 4: Example of one-dimensional classical Christoffel-Darboux sum for orthonormal Chebyshev polynomials of the first kind for odd order n, n=15, in the normalized interval

3 A novel two-dimensional continuous Christoffel-Darboux formula for Chebyshev polynomials of the first kind

The proposed two-dimensional continuous symmetric Christoffel-Darboux formula for orthonormal Chebyshev polynomials of the first kind is the generalization of the results of one-dimensional function $\Phi_n(x)$, and it is given by the formula 5. The new formula $\Phi_n(x,y)$ is defined for two real variables as

$$\begin{aligned} \Phi_n(x,y) &= \frac{T_0(x)T_0(x)\cdot T_0(y)T_0(y)}{h_0h_0} + \\ &+ \frac{T_1(x)T_1(x)\cdot T_1(y)T_1(y)}{h_1h_1} + \\ &\vdots \\ &+ \frac{T_{n-1}(x)T_{n-1}(x)\cdot T_{n-1}(y)T_{n-1}(y)}{h_{n-1}h_{n-1}} + \\ &+ \frac{T_n(x)T_n(x)\cdot T_n(y)T_n(y)}{h_nh_n} \end{aligned} \tag{9}$$

i. e.

$$\begin{aligned} \Phi_n(x,y) &= \frac{T_0(x)T_0(x)\cdot T_0(y)T_0(y)}{\pi\cdot\pi} + \\ &+ \frac{T_1(x)T_1(x)\cdot T_1(y)T_1(y)}{\frac{\pi}{2}\cdot\frac{\pi}{2}} + \\ &\vdots \\ &+ \frac{T_{n-1}(x)T_{n-1}(x)\cdot T_{n-1}(y)T_{n-1}(y)}{\frac{\pi}{2}\cdot\frac{\pi}{2}} + \\ &+ \frac{T_n(x)T_n(x)\cdot T_n(y)T_n(y)}{\frac{\pi}{2}\cdot\frac{\pi}{2}} \end{aligned} \tag{10}$$

$$\left(\frac{\pi}{2}\right)^2 \Phi_n(x,y) = \begin{aligned} & \frac{1}{4}\cdot T_0(x)T_0(x)\cdot T_0(y)T_0(y) \quad + \\ & + \quad T_1(x)T_1(x)\cdot T_1(y)T_1(y) \quad + \\ & \vdots \\ & + \quad T_{n-1}(x)T_{n-1}(x)\cdot T_{n-1}(y)T_{n-1}(y) \quad + \\ & + \quad T_n(x)T_n(x)\cdot T_n(y)T_n(y) \end{aligned} \tag{11}$$

$$\left(\frac{\pi}{2}\right)^2 \Phi_n(x,y) = \frac{1}{4}\cdot T_0(x)T_0(x)\cdot T_0(y)T_0(y) + \sum_{r=1}^{n} T_r(x)T_r(x)\cdot T_r(y)T_r(y) \tag{12}$$

In Table 2 a two-dimensional function Christoffel-Darboux formula for orthonormal Chebyshev polynomials of the first kind, for polynomials order $n = 1, 2, \ldots, 7$, is given.

In the theory of electrical filters the normalized function, $\Phi_{n,nor}(x,y)$, is applied in the following form:

$$\Phi_{n,nor}(x,y) = \frac{\Phi_n(x,y) - \Phi_n(0,0)}{\Phi_n(1,1) - \Phi_n(0,0)} \tag{13}$$

Figures 5, 6 and 7 illustrate the examples of odd order (*n=11*) of proposed two-dimensional Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the first kind.

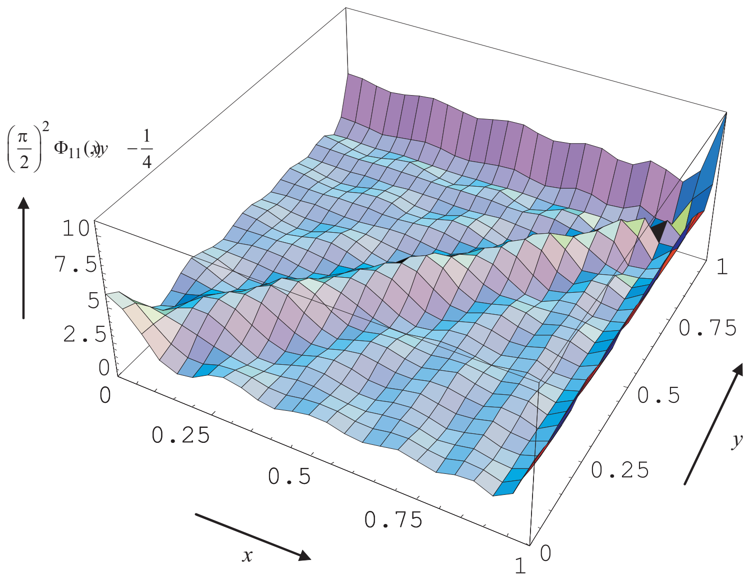

Fig. 5: 3D plot of the proposed symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the first kind odd order, n=11

Table 2: CONTINUOUS TWO-DIMENSIONAL CHRISTOFFEL-DARBOUX FORMULA FOR ORTHONORMAL CHEBYSHEV POLYNOMIALS OF THE FIRST KIND, FOR DIFFERENT VALUES OF n

n	$\left(\frac{\pi}{2}\right)^2 \Phi_n(x,y) - \frac{1}{4} = \sum_{r=1}^{n} T_r(x)T_r(x) \cdot T_r(y)T_r(y)$
1	y^2x^2
2	$1-4y^2+4y^4-4x^2+17y^2x^2-16y^4x^2+4x^4-16y^2x^4+16y^4x^4$
3	$1-4y^2+4y^4x^2+98y^2x^2-232y^4x^2+144y^6x^2+4x^4-232y^2x^4$ $+592y^4x^4-384y^6x^4+144y^2x^6-384y^4x^6+256y^6x^6$
4	$2-20y^2+84y^4-128y^6+64y^8-20x^2+354y^2x^2-1512y^4x^2$ $+2192y^6x^2-1024y^8x^2+84x^4-1512y^2x^4+6992y^4x^4-10624y^6x^4$ $+5120y^8x^4-128x^6+2192y^2x^6-10624y^4x^6+16640y^6x^6$ $-8192y^8x^6+64x^8-1024y^2x^8+5120y^4x^8-8192y^6x^8+4096y^8x^8$
5	$2-20y^2+84y^4-128y^6+64y^8-20x^2 979y^2x^2-6512y^4x^2+16192y^6x^2$ $-17024y^8x^2+6400y^{10}x^2+84x^4-6512y^2x^4+46992y^4x^4-122624y^6x^4$ $+133120y^8x^4-51200y^{10}x^4-128x^6+16192y^2x^6-122624y^4x^6+330240y^6x^6$ $-366592y^8x^6+143360y^{10}x^6+64x^8-17024y^2x^8+133120y^4x^8-366592y^6x^8$ $+413696y^8x^8-163840y^{10}x^8+6400y^2x^{10}-51200y^4x^{10}+143360y^6x^{10}$ $-163840y^8x^{10}+65536y^{10}x^{10}$
6	$3-56y^2+504y^4-1920y^6+3520y^8-3072y^{10}+1024y^{12}-56x^2+2275y^2x^2$ $-21632y^4x^2+80704y^6x^2-141440y^8x^2+116992y^{10}x^2-36864y^{12}x^2$ $+504x^4-21632y^2x^4+223392y^4x^4-875264y^6x^4+1584640y^8x^4$ $-1341440y^{10}x^4+430080y^{12}x^4-1920x^6+80704y^2x^6-875264y^4x^6$ $+3541504y^6x^6-6559744y^8x^6+5648384y^{10}x^6-1835008y^{12}x^6$ $+3520x^8-141440y^2x^8+1584640y^4x^8-6559744y^6x^8+12357632y^8x^8$ $-10780672y^{10}x^8+3538944y^{12}x^8-3072x^{10}+116992y^2x^{10}-1341440y^4x^{10}$ $+5648384y^6x^{10}-10780672y^8x^{10}+9502720y^{10}x^{10}-3145728y^{12}x^{10}$ $+1024x^{12}-36864y^2x^{12}+430080y^4x^{12}-1835008y^6x^{12}+3538944y^8x^{12}$ $-3145728y^{10}x^{12}+1048576y^{12}x^{12}$
7	$3-56y^2+504y^4-1920y^6+3520y^8-3072y^{10}+1024y^{12}-56x^2+4676y^2x^2$ $-60048y^4x^2+311200y^6x^2-800000y^8x^2+1082880y^{10}x^2-739328y^{12}x^2$ $+200704y^{14}x^2+504x^4-60048y^2x^4+838048y^4x^4-4563200y^6x^4$ $+12121600y^8x^4-16795648y^{10}x^4+11669504y^{12}x^4-3211264y^{14}x^4-1920x^6$ $+311200y^2x^6-4563200y^4x^6+25669120y^6x^6-69781504y^8x^6$ $+98373632y^{10}x^6-69271552y^{12}x^6+19267584y^{14}x^6+3520x^8-800000y^2x^8$ $+12121600y^4x^8-69781504y^6x^8+192991232y^8x^8-275709952y^{10}x^8$ $+196214784y^{12}x^8-55050240y^{14}x^8-3072x^{10}+1082880y^2x^{10}-16795648y^4x^{10}$ $+98373632y^6x^{10}-275709952y^8x^{10}+398065664y^{10}x^{10}-285736960y^{12}x^{10}$ $+80740352y^{14}x^{10}+1024x^{12}-739328y^2x^{12}+11669504y^4x^{12}-69271552y^6x^{12}$ $+196214784y^8x^{12}-285736960y^{10}x^{12}+206569472y^{12}x^{12}-58720256y^{14}x^{12}$ $+200704y^2x^{14}-3211264y^4x^{14}+19267584y^6x^{14}-55050240y^8x^{14}$ $+80740352y^{10}x^{14}-58720256y^{12}x^{14}+16777216y^{14}x^{14}$

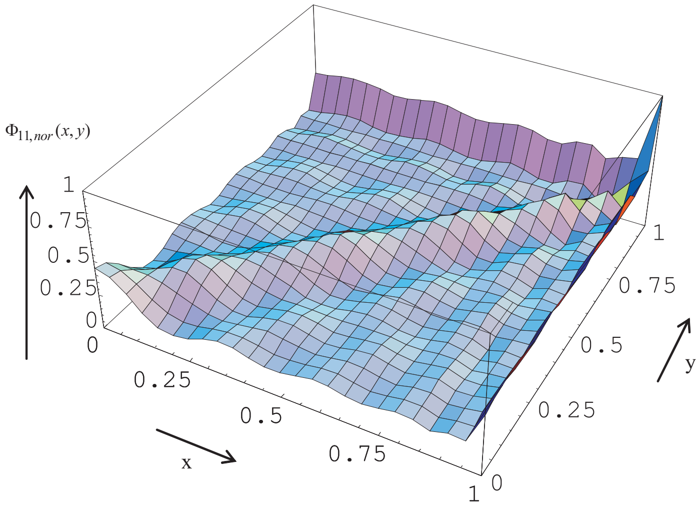

Fig.6: 3D plot of the proposed normalized symetric two-dimensinal continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the first kind odd order, n=11

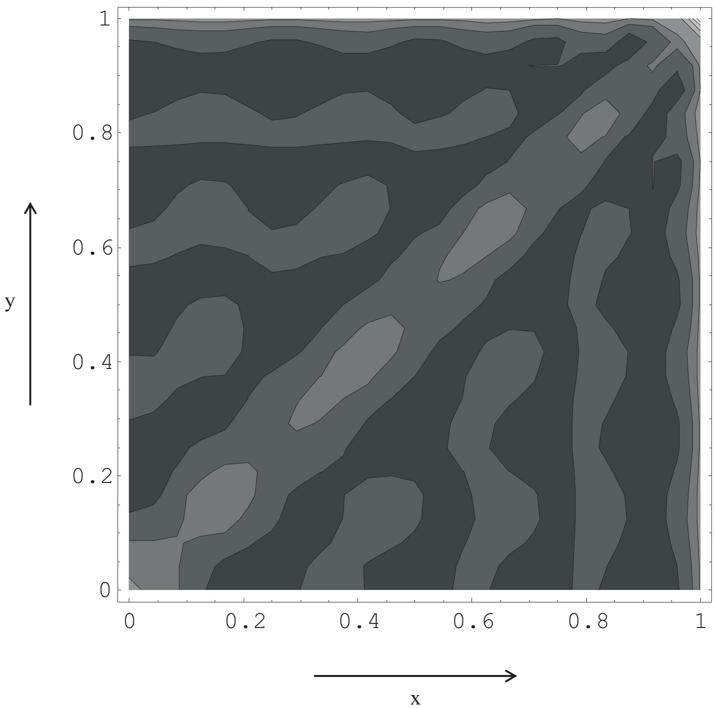

Fig. 7: 2D contour plot of the proposed symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the first kind odd order, n=11

Figures 8, 9 and 10 illustrate the examples of even order (n=12) proposed two-dimensional Christoffel-Darboux formula for orthonormal Chebyshev polynomials of the first kind.

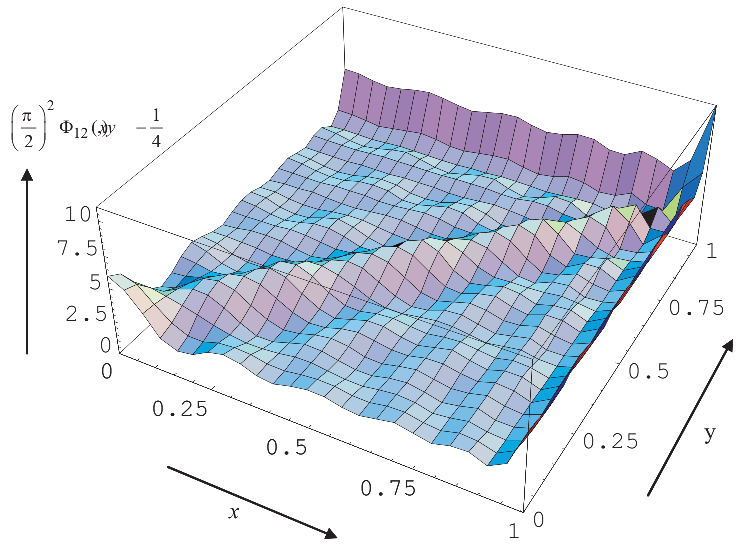

Fig. 8: 3D plot of the proposed symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the first kind even order, n=12

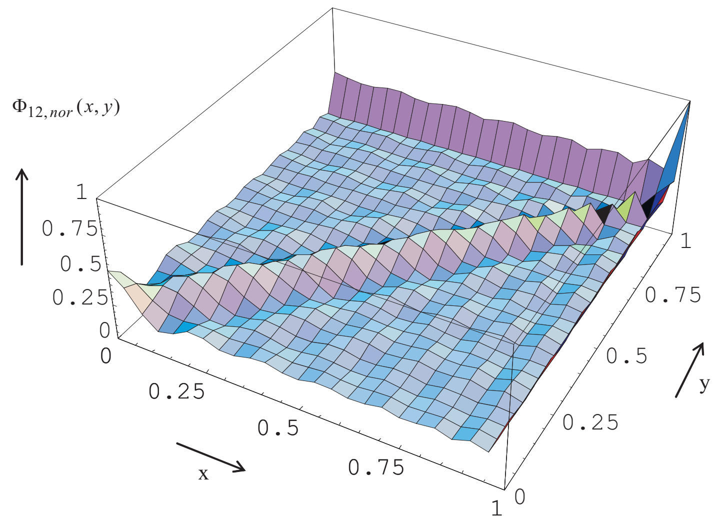

Fig. 9: 3D plot of the proposed normalized symetric two-dimensinal continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the first kind odd order, n=12

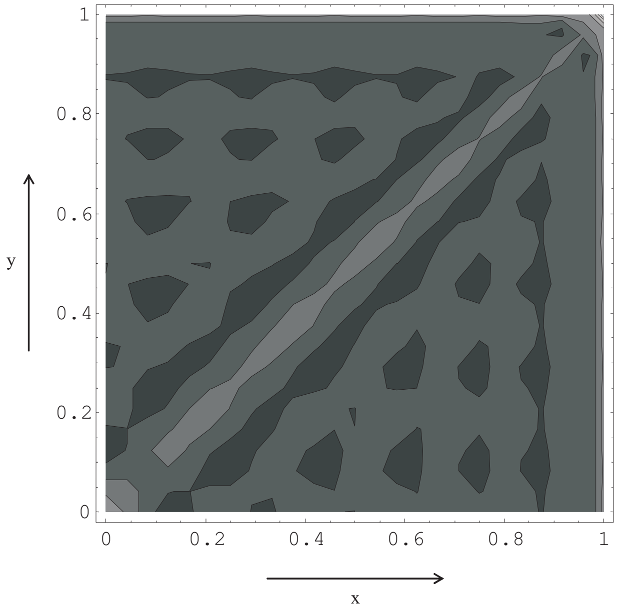

Fig. 10: 2D contour plot of the proposed symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the first kind even order, n=12

4 Conclusion

Symmetric two-dimensional continual Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the first kind in compact explicit form is presented in this paper. The formula is general and applies to the even and odd order for two real variables.

The examples of even and odd order of the proposed formula are illustrated and shown in the table of two-dimensional real function, $\Phi_n(x,y)$, which is generated by the proposed formula, and is applicable in the technique for the synthesis of analog and digital one- and two-dimensional filter functions.

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