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Contribution to the Orthogonal Projection of H_3 Space onto Orisphere and Introduction of Distance Method

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Abstract: In this paper we shall present, like in Euclidean space, in hyperbolic space H_3 all spatial objects by projecting them in the plane of image by the use of some mapping methods. In that case we shall solve the space problems by solving the corresponding problems in a plane. We shall present some proofs, different from the proofs given by Z. A. Skopec [4], [2], related to the characteristics of projecting H_3 space onto orisphere. The advantage of Skopec's method [5] will be also emphasized. By the use of analogy with the Euclidean space we shall define, in H_3 space, the distance method.

Keywords: equidistant curve, oricircle, orisphere.

1 Introduction

In the first half of XIX century Russian mathematician N. I. Lobachevsky solved one difficult and centuries-long problem related to the independence of axiom of parallelism from the other Euclidean geometry axioms. It strongly influenced the further and even today's development of contemporary mathematics.

In hyperbolic H_2 plane there are three types of pencils of straight lines:

1. Elliptic, Fig. 1 a);
2. Hyperbolic, Fig. 1 b);
3. Parabolic, Fig. 1 c).

Orthogonal trajectories of these pencil lines are:

1') Circles, Fig. 1 a.);

2') Equidistant curve, Fig. 1 b);

3') Oricircle, Fig. 1 c).

By an analogy, in H_3 hyperbolic space there are three types of pencil lines

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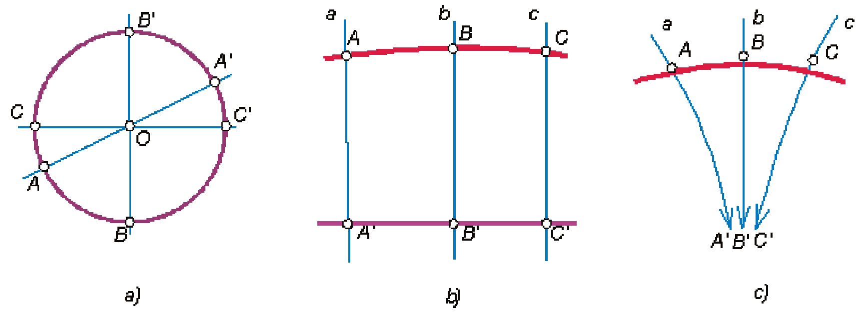

Fig. 1. Pencil of straight lines in H_2 plane and their orthogonal trajectories (circle (a); equidistant line (b); oricircle (c))

I) Elliptic, II) Hyperbolic, III) Parabolic.

In H_3 hyperbolic space to the above mentioned three types of pencil lines correspond three types of surfaces, as their orthogonal trajectories:

I') Sphere, II') Equidistant surface, III') Orisphere.

One of the well known interpretations of hyperbolic geometry is the interpretation of A. Poencar which we shall describe shortly in the case of H_2. For this purpose we shall consider some straight line in $E2$ Euclidean plane which could coincide with the axis x, without diminishing generality.

The points of upper half-plane we consider as hyperbolic points and the whole upper half-space as hyperbolic H_2 plane. The points on x axis are not hyperbolic points. The straight lines of hyperbolic H_2 plane are considered as semicircles in the upper half-space, orthogonal to x axis, including semicircles with infinite radius.

We consider two figures in H_2 as equivalent or congruent if there is finite number of transforms $f_1,\ldots\ldots f_n$, each of which is the inversion with the center on x axis, and transform $f = f_n * f_{n-1} * \ldots\ldots * f_1$ is transforming one of given figures into the other.

In this model it was shown that all axioms of hyperbolic geometry H_2 are valid. For example, from Fig. 2 it is evident that according to Poencar's model the Lobachevsky axiom is fulfilled.

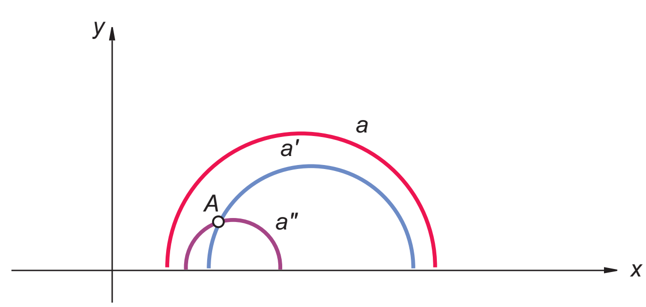

Fig. 2. Interpretation of Lobachevsky's axiom

2 Contribution to the projection of H_3 space onto orisphere

It has to be emphasized that the interior of sphere geometry is usually underlined as a model of two dimensional elliptic geometry. However, it is possible to choose the objects of sphere in a such way to achieve interior geometry which is the model of two dimensional euclidean geometry. Also, in [3] is given the choice of euclidean geometry on the sphere by which the orisphere is presented in Poencare's model of hyperbolic space. This fact initiates the projection of H_3 onto orisphere.

We shall carry out the projection of H_3 onto orisphere G by the pencil of straight lines parallel in one direction and which are the axes of orisphere. We shall consider that the orisphere is supplemented with one faraway point which belongs to the parallel axes of orisphere.

If A is an arbitrary point of H_3 then we have:

Definition 1 *The point A_0 is called an orthogonal projection of point A on orisphere G in H_3 space in which the orisphere axis through A penetrates orisphere (Fig. 3). Further on we shall formulate some theorems and prove them in a different way than as it was done in [4].*

Further on we shall formulate some theorems and prove them in a different way as was done in [4].

Theorem 1 *To each point of H_3 correspond on orisphere only one point as its orthogonal projection on orisphere. The opposite is not true.*

Proof. Through point A (Fig. 3a) there is only one orisphere axis a and only one its penetrating point through orisphere which is the orthogonal projection A_0 of point A. The opposite is not true as to an arbitrary point on orisphere G, for example point A_0, chosen as the projection point of H_3 space, corresponds in H_3 each point on axis a through point A_0 on orisphere G. Therefore, for locating the position of some point in H_3 it is not sufficient to know only its orthogonal projection on G. To locate unambiguously the position of point A in H_3 space it is necessary to know except the orthogonal projection A_0 on G also the distance AA_0 which is called the distance of point A from the orisphere G.

Definition 2 *The set of penetrating points produced by projective rays of the straight line q which are orisphere axes is the projection of straight line q. The point S of straight line q in which the orisphere axis s, which is orthogonal to q, intersects q is named center of straight line q. If the point S_0 is the projection of point S, then the segment SS_0 is called the distance of line q from the orisphere.*

Theorem 2 *To each straight line q which is not the orisphere axis, corresponds the unique pair of points $(Q', Q")$ onto orisphere and vice versa (Fig. 3b).*

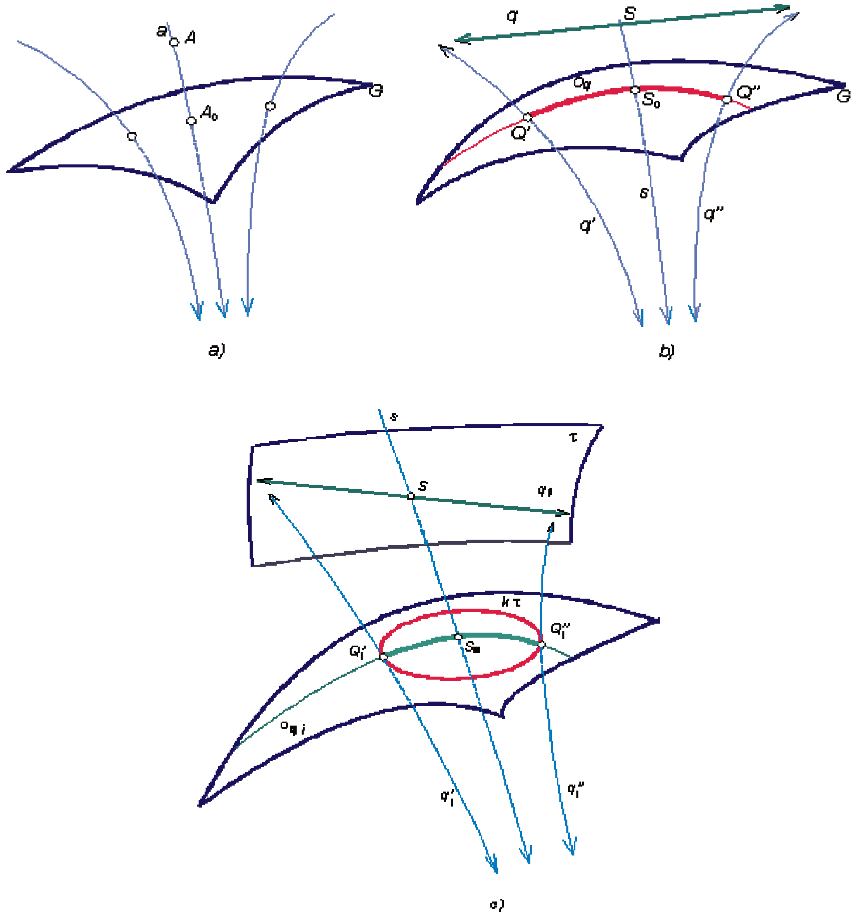

Fig. 3. The projections of objects onto orisphere (point (a), straight line (b), plane (c))

Proof. Having in mind that the projective rays of the straight line q are the axes of orisphere G, then they belong to one plane and the projection of the straight line **q** belongs to oricircle o_q which represents intersection of the projection plane of straight line q with the orisphere G. The projection points of straight line q determine the oricircle o_q with the endpoints Q' and Q''. Through Q' and Q'' pass the axes q' and q'' of orisphere parallel with line q, one of them is in one direction and the other one is in other direction. The arc $Q'Q''$ of oricircle o_q determines the axes q' and q'' of orisphere G and they determine only one straight line q. The arc length Q”Q”” of oricircle o_q depends on the distance between the straight line q and orisphere G. When the projection S_0 on o_q and the distance SS_0 of center S are given, then the line q is determined on o_q in H_3 and therefore the axes q' and q'' of the orisphere are determined as well as end points Q' and Q'' on o_q.

At orthogonal projection of H_3 on G to the straight line, which is the axis of orisphere, we correspond the pair of points one of which is the point where q penetrates through orisphere and the other one is infinitely faraway with which we supplemented G. As a corollary of previous theorem we have:

Corollary 1 *To each straight line q in $H3$, which is not axis of orisphere G, corresponds on G unique oricircle on which it is determined one point S_0 and its distance SS_0 from G, where S is the center of straight line q and vice versa*

Definition 3 *The projection of plane τ on orisphere G is the set of the projections of their points. The point S of plane $\top$ is the center of plane $\top$ if S is the penetration point of axis s that is orthogonal to $\top$. If $\mathbf{S_0}$ is the projection of point S on G then the segment SS_0 is called* <u>*the distance*</u> *of plane $\top$ from G.*

Theorem 3 *To each plane, which does not contain any axis of orisphere G, when it is orthogonally projected on G, corresponds only one circle and vice versa.*

<u>Proof.</u> Let $\top$ is an arbitrary plane in H_3 space which does not contain any axis of orisphere G,S- the center of plane $\top$ and $S(q_i)$- eliptic pencil of straight lines q_i in plane $\top$.

As S is the center of plane $\top$ then S is also the center of each straight line q_i in the pencil of straight lines $S(q_i)$. Therefore S_0 is the center of each arc $Qi'Qi''$ of oricircle o_{qi} at which the projected plane μ_i of line q_i intersects G. As all lines from the pencil straight lines $S(q_i)$ have the same distance then all arcs are equal. Therefore, the points $Qi'Qi'$ of orisphere G belong to circle $K_\top$ (Fig.3c).

As all points of straight line q_i are projected into interior points of an arc $Qi'Qi''$ of oricircle o_{qi}, then all points of plane $\top$ are projected to the points inside the circle $K_\top$ on orisphere. Also, to the circle $K_\top$ of orisphere G corresponds completely determined plane $\top$ in $H3$ space. The radius of the circle $K_\top$ depends on SS_0-the distance between plane $intercal$ and the orisphere G. In a special case, if the plane contains the axis of orisphere then to it corresponds the oricircle as an intersection of orisphere G and that plane.

According to this theorem we have the following corollary:

Corollary 2 *To an arbitrary plane $intercal$ in H_3 space which does not contain any axis of orisphere G corresponds the unique circle $K_\top$ on* **G** *with the determined center point in S_0 , its distance from G is SS_0 and S is the center of plane $intercal$. According to definitions: 1, 2, 3 and theorems: 1, 2, 3 by the orthogonal projection of H_3 space on orisphere G, the point corresponds to the point, the straight line corresponds to the pair of points (the end points of an oricicle arc), and the plane corresponds to the circle, i.e. to the points of orisphere bordered by circle or oricircle, and vice versa. The arc of oricircle with the end points Q' and Q"; the points of orisphere inside the circle $K_\top$ together with the circle $K_\top$, are called* <u>*respectively complete projections of line*</u> *or* <u>*complete projection of plane.*</u>

The point A which belongs to the straight line a we designate with $A(A_0,A',A")$. The straight line q to which corresponds the pair of points (Q',Q") on the orisphere we designate with $q(Q',Q")$. The plane $\top$ to which corresponds the circle $K_\top$ on orisphere we designate with $\top$ $(K_\top)$.

Having in mind that the interior geometry of orisphere is euclidean, then the orisphere, supplemented by one infinitely faraway point, to which we projected H_3 space together with all their points, oricircles and circles we may consider as **Mobius** planes with their points, straight lines and circles. The projections of spatial objects are in this plane which is therefore called image plane.

According to previously stated theorems and definitions it is possible to prove the following theorems related to respective positions of points, straight lines and planes in $H3$ space.

Theorem 4 *The straight line $a(A'A")$ belongs to the plane $\top(K_\top)$ if and only if the points A' and $A"$ belong to the circle $K_\top$*

Theorem 5 *Two planes $\top(K_\top)$ and $\mu(K_\mu)$ are crossing each other, they are parallel or bypassing each other if and only if the circles $K_\top$ and K_μ are crossing, touching or have not common points, respectively.*

Theorem 6 *The straight line $p(P',P")$ is perpendicular to the plane $\top(K_\top)$ if and only if the points P' and P'' are inverse points respectively to $K_\top$.*

3 The distance method in $\mathbf{H_3}$

According to definitions: 1, 2, and 3 as well as to the corollaries 1 and 2 we conclude that during the orthogonal projections of hyperbolic space H_3 on the orisphere G very important role have distances of points, straight lines and planes from the orisphere.

So we have that:

a) The arc length of oricircle in which the straight line is projected depends on straight line distance from orisphere.

b) The radius of circle, that confines the points of orisphere to which the plane is projected, depends on distance of plane from the orisphere.

c) According to corollary 1, the plane q of $H3$ space can be specified by one oricircle o_q, the point S_0 of this oricircle and by the distance SS_0, where S is the center of straight line. Then the corresponding designation for the straight line is $q(q', S_0, S_0S)$.

d) Completely analogous, according to corollary 2 the corresponding designation for the plane $\top$ is $\top(\top', S_0, S_0S)$.

According to a), b), c), d) we see that for presenting the points, straight lines and planes of $H3$ space on the orisphere an important role have the distances from the orisphere and that is the reason to call this method the distance method.

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