New 2D Continuous Symmetric Christoffel-Darboux Formula for Chebyshev Orthonormal Polynomials of the Second Kind

E. Karoussos, V. D. Pavlović, J. R. Djordjević-Kozarov, Č. B.Doličanin

Abstract: In this paper, we propose a new two-dimensional continuous symmetric Christoffel-Darboux formula for orthonormal classical Chebyshev polynomials of the second kind. This continuous two-dimensional function of two real variables is most directly applied to approximation problems and synthesis of filter functions. The examples of the proposed two-dimensional Christoffel-Darboux formula are illustrated.

Keywords: Christoffel-Darboux formula, Chebyshev polynomials of the second kind, two-dimensional functions, classical orthogonal functions

1 Introduction

Implementation of classical orthogonal polynomials in filter theory is given in [1-18]. Complete theory of one-dimensional classical continuous orthogonal polynomials is described in great literature [19 - 25]. Wide range of application of extremal properties of Christoffel-Darboux formula in physics is described in the literature [26 - 31].

A capital new originally general solution of approximation problem as a prototype of all-pole low-pass continual time filter functions is proposed in this paper. Approximation of ideal filter function is derived using Christoffel-Darboux formula for the set of continual Jacobi orthogonal polynomials on the finite interval [-1, +1], with respect to the continuous weight function involving the couple of real free parameters, \( \alpha \) and \( \beta \).

Many problems in continuous or discrete domain can be solved by applying the extremal properties of Christoffel-Darboux sum.

Many actual design problems can be solved by choosing a filter function as result of the approximation technique by orthogonal polynomials known as the classical, as they are: Gegenbauer (ultraspherical), Chebyshev (first and second kind) and Legendre (spherical) polynomials. Some of applications of orthogonal polynomials in theory of electrical filters are considered in papers [1-18]. The Jacobi orthogonal polynomials, \( P_n^{(\alpha, \beta)}(x) \), hold a key role in such problems.

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position in a hierarchy of orthogonal polynomial classes. For certain choices of the real parameter values, $\alpha$ and $\beta$, several classes of orthogonal polynomials considered "classical" were produced as special or limiting cases of the Jacobi polynomials class. Therefore, finding a filter function class attributed to Jacobi orthogonal polynomials was a challenging task. In this paper proposed approximation inherit the extremal property of the parent Jacobi orthogonal polynomials and demonstrate their suitability in modern filter designing applications.

By solving it task we found that derived approximation, which proposed in this work, can be used to generate set of new filter function and set of particular solutions wich generate in literature well known classical filter functions.

In this paper, we propose an original two-dimensional symmetric Christoffel-Darboux formula for classical orthonormal Chebyshev polynomials of the second kind in the compact explicit form, which is valid for even and odd order. The examples are illustrated and tables of two-dimensional symmetric functions of low order are given.

## 2 One-dimensional classical Christoffel-Darboux formula for orthonormal Chebyshev polynomials of the second kind

The general form of a polynomial function of n-th order is given in the following expression:

$$\phi_n(x) = \sum_{k=0}^{n} a_k x^k$$ (1)

Chebyshev orthogonal polynomials of n-th order of the second kind, $U_n(x)$, the real variable $x$, are given in the following expression:

$$U_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2x)^{n-2k}$$ (2)

Chebyshev orthogonal polynomials are orthogonal on interval [-1, 1] with the weighting function $\omega(x)$

$$\omega(x) = \sqrt{1-x^2}$$ (3)

Where the norm is

$$h_r = \int_{-1}^{1} \sqrt{1-x^2} \ U_r(x) \ U_r(x) \ dx = \begin{cases} \pi/2, & r \neq 0 \end{cases}$$ (4)

**Example 1:**

For even $n, n = 12$, (i.e. $n = 2r$) Chebyshev polynomials of the second kind, $U_{12}(x)$, has the form

$$U_{12}(x) = 1 - 84 x^2 + 1120 x^4 - 5376 x^6 + 11520 x^8 - 11264 x^{10} + 4096 x^{12}$$

and it is shown in Fig. 1.
New 2D continuous symmetric Christoffel-Darboux formula ...

Example 2:
For odd \( n, n = 15 \), (i.e. \( n = 2r + 1 \)) Chebyshev polynomials of the second kind, \( U_{15}(x) \), has the form
\[
U_{15}(x) = -16x + 672x^3 - 8064x^5 + 42240x^7 - 112640x^9 + 159744x^{11} - 114688x^{13} + 32768x^{15}
\]
and it is shown in Fig. 2.

Extremal properties of the polynomial continuous classical one-dimensional real function is given by Christoffel-Darboux formula for Chebyshev orthogonal polynomials of the second kind and has the form:

\[
\Phi_n(x) = \frac{[U_0(x)]^2}{\frac{x}{2}} + \frac{[U_1(x)]^2}{\frac{x}{2}} + \frac{[U_2(x)]^2}{\frac{x}{2}} + \ldots + \frac{[U_{n-1}(x)]^2}{\frac{x}{2}} + \frac{[U_n(x)]^2}{\frac{x}{2}} \quad (5)
\]

Based on the expression (5), Table 1 shows examples of polynomial functions of Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind for different values.

Based on equality (5) fundamental observations were performed in [20], known in the literature as Bessel’s and Parserval’s formula.
Table 1: One-dimensional classical Christoffel-Darboux formula for the given order $n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\Phi_n(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1$</td>
</tr>
<tr>
<td>1</td>
<td>$1 + 4x^2$</td>
</tr>
<tr>
<td>2</td>
<td>$2 - 4x^2 + 16x^4$</td>
</tr>
<tr>
<td>3</td>
<td>$2 + 12x^2 - 48x^4 + 64x^6$</td>
</tr>
<tr>
<td>4</td>
<td>$3 - 12x^2 + 128x^4 - 320x^6 + 256x^8$</td>
</tr>
<tr>
<td>5</td>
<td>$5 + 24x^2 - 256x^4 + 1088x^6 + 1792x^8 + 1024x^{10}$</td>
</tr>
<tr>
<td>6</td>
<td>$4 - 24x^2 + 480x^4 - 2880x^6 + 7680x^8 - 9216x^{10} + 4096x^{12}$</td>
</tr>
<tr>
<td>7</td>
<td>$4 + 40x^2 - 800x^4 + 6592x^6 - 25088x^8 + 48128x^{10} + 45056x^{12} + 16384x^{14}$</td>
</tr>
<tr>
<td>8</td>
<td>$5 - 40x^2 + 1280x^4 - 13504x^6 + 68864x^8 - 187392x^{10} + 278528x^{12} - 212992x^{14} + 65536x^{16}$</td>
</tr>
<tr>
<td>9</td>
<td>$5 + 60x^2 - 1920x^4 + 25536x^6 - 166656x^8 + 602112x^{10} - 1261568x^{12} + 1523712x^{14} - 983040x^{16} + 262144x^{18}$</td>
</tr>
<tr>
<td>10</td>
<td>$6 - 60x^2 + 2800x^4 - 45248x^6 + 36592x^8 - 1683456x^{10} + 4653056x^{12} - 7880704x^{14} + 7995392x^{16} - 4456448x^{18} + 1048576x^{20}$</td>
</tr>
<tr>
<td>11</td>
<td>$684x^2 - 3920x^4 + 76160x^6 - 747520x^8 + 4231168x^{10} - 14778368x^{12} + 32849920x^{14} - 46530560x^{16} + 40632320x^{18} - 19922944x^{20} + 4194304x^{22}$</td>
</tr>
<tr>
<td>12</td>
<td>$7 - 84x^2 + 5376x^4 - 122752x^6 + 1433088x^8 - 9768960x^{10} + 41828352x^{12} - 116932608x^{14} + 216465408x^{16} - 262930432x^{18} + 201326592x^{20} - 88080384x^{22} + 16777216x^{24}$</td>
</tr>
<tr>
<td>13</td>
<td>$7 + 112x^2 - 7168x^4 + 190848x^6 - 2609664x^8 + 21039104x^{10} - 107954176x^{12} + 368328704x^{14} - 854130688x^{16} + 1351090176x^{18} - 1434451968x^{20} + 972727832x^{22} - 385875968x^{24} + 67108864x^{26}$</td>
</tr>
<tr>
<td>14</td>
<td>$8 - 112x^2 + 9408x^4 - 287616x^6 + 4549632x^8 - 42747904x^{10} + 258236416x^{12} - 1051541504x^{14} + 2965110784x^{16} - 858772672x^{18} + 807193048x^{20} - 7604273152x^{22} + 4664606048x^{24} - 1677721600x^{26} + 268435456x^{28}$</td>
</tr>
</tbody>
</table>

3 A novel two-dimensional continuous Christoffel-Darboux formula for Chebyshev polynomials of the second kind

The proposed two-dimensional continuous symmetric Christoffel-Darboux formula for orthonormal Chebyshev polynomials of the second kind is the generalization of the results of $\Phi_n(x,y)$, and it is given by the formula (5). The new formula $\Phi_n(x,y)$ is defined for two real variables as

$$\Phi_n(x,y) = \frac{U_n(x)U_n(y)}{\sqrt{h_0h_2h_2h_0}} + \frac{U_{n-1}(x)U_{n-1}(y)}{\sqrt{h_1h_1h_1h_1}} + \cdots + \frac{U_{n-r+1}(x)U_{n-r+1}(y)}{\sqrt{h_{n-r+1}h_{n-r+1}h_{n-r+1}}} + \frac{U_{n-r}(x)U_{n-r}(y)}{\sqrt{h_nh_nh_nh_n}}$$

respectively

$$\Phi_n(x,y) = \sum_{r=1}^{n} \frac{U_r(x)U_r(y)}{(\frac{n}{2})^2}$$
Homogeneous part of function, $\Phi(x) - \Phi(0)$, is of interest for solving problems in engineering, physics and atomic physics, and it is presented in the form for two real variables

$$\Phi_n(x,y) - \Phi_n(0,0)$$

or normalized form of the homogeneous part of function

$$\frac{\Phi_n(x,y) - \Phi_n(0,0)}{\Phi_n(1,1) - \Phi_n(0,0)}$$

In Table 2 a two-dimensional function Christoffel-Darboux formula for orthonormal Chebyshev polynomials of the second kind, for polynomials order $n = 1, 2, \ldots, 7$, is given.

For even order, $n = 12$, the Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind $\Phi_{12}(x)$ is illustrated in Fig. 3.

For odd order, $n = 15$, Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind $\Phi_{15}(x)$ is illustrated in Fig. 4.

Figures 5, 6 and 7 illustrate the examples of odd order ($n = 11$) of proposed two-dimensional Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind.

Figures 8, 9 and 10 illustrate the examples of even order ($n = 16$) proposed two-dimensional Christoffel-Darboux formula for orthonormal Chebyshev polynomials of the second kind.
Table 2: Continuous two-dimensional Christoffel-Darboux formula for orthonormal Chebyshev polynomials of the second kind, for different values of \( n \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( (\frac{1}{2})^n \Phi_n(x,y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 16y^2x^2 )</td>
</tr>
<tr>
<td>2</td>
<td>( 1 - 8y^2 + 16y^4 - 8x^2 + 80y^2x^2 - 128y^4x^2 + 16x^6 - 128y^6x^2 + 256y^8x^4 )</td>
</tr>
<tr>
<td>3</td>
<td>( 1 - 8y^2 + 16y^4 - 8x^2 + 336y^2x^2 - 1152y^4x^2 + 1024y^6x^2 + 16x^6 - 1152y^6x^2 )</td>
</tr>
<tr>
<td>4</td>
<td>( 2 - 32y^2 + 192y^4 - 384y^6 + 256y^8 - 32x^2 + 912y^2x^2 - 5376y^4x^2 + 1024y^6x^2 )</td>
</tr>
<tr>
<td>5</td>
<td>( -6144y^4x^2 + 192x^4 - 5376y^2x^4 - 35328y^4x^4 - 71680y^6x^4 + 45056y^8x^4 - 384x^8 )</td>
</tr>
<tr>
<td>6</td>
<td>( +1024x^6 - 71680y^2x^6 + 151152y^4x^6 - 98304y^6x^6 + 256x^8 - 6144y^2x^8 )</td>
</tr>
<tr>
<td>7</td>
<td>( +45056y^4x^8 - 98304y^6x^8 + 65536y^8x^8 )</td>
</tr>
</tbody>
</table>

\[ 2 - 32y^2 + 192y^4 - 384y^6 + 256y^8 - 32x^2 + 912y^2x^2 - 5376y^4x^2 + 1024y^6x^2 \]

\[ -6144y^4x^2 + 192x^4 - 5376y^2x^4 - 35328y^4x^4 - 71680y^6x^4 + 45056y^8x^4 - 384x^8 \]

\[ +1024x^6 - 71680y^2x^6 + 151152y^4x^6 - 98304y^6x^6 + 256x^8 - 6144y^2x^8 \]

\[ +45056y^4x^8 - 98304y^6x^8 + 65536y^8x^8 \]
New 2D continuous symmetric Christoffel-Darboux formula ...

Fig. 4: For odd order, $n = 15$, the Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind $\Phi_{15}(x)$

Fig. 5: 3D plot of the proposed symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind odd order, $n=11$

Fig. 6: 3D plot of the proposed normalized symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind odd order, $n=11$
Fig. 7: 2D contour plot of the proposed symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind odd order, n=11

Fig. 8: 3D plot of the proposed symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind even order, n=16
New 2D continuous symmetric Christoffel-Darboux formula ...

Fig. 9: 3D plot of the proposed normalized symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind even order, \( n=16 \).

Fig. 10: 2D contour plot of the proposed symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind even order, \( n=16 \).
4 Conclusion

Symmetric two-dimensional continual Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind in compact explicit form is presented in this paper. The formula is general and applies to the even and odd order for two real variables.

The examples of even and odd order of the proposed formula are illustrated and shown in the table of two-dimensional real function, $\Phi_n(x,y)$, which is generated by the proposed formula, and is applicable in the technique for the synthesis of analog and digital filter functions.

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References


