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New 2D Continuous Symmetric Christoffel-Darboux Formula for Chebyshev Orthonormal Polynomials of the Second Kind

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Abstract: In this paper, we propose a new two-dimensional continuous symmetric Christoffel-Darboux formula for orthonormal classical Chebyshev polynomials of the second kind. This continuous two-dimensional function of two real variables is most directly applied to approximation problems and synthesis of filter functions. The examples of the proposed two-dimensional Christoffel-Darboux formula are illustrated.

Keywords: Christoffel-Darboux formula, Chebyshev polynomials of the second kind, two-dimensional functions, classical orthogonal functions

1 Introduction

Implementation of classical orthogonal polynomials in filter theory is given in [1-18]. Complete theory of one-dimensional classical continuous orthogonal polynomials is described in great literature [19 - 25]. Wide range of application of extremal properties of Christoffel-Darboux formula in physics is described in the literature [26 - 31].

A capital new originally general solution of approximation problem as a prototype of all-pole low-pass continual time filter functions is proposed in this paper. Approximation of ideal filter function is derived using Christoffel-Darboux formula for the set of continual Jacobi orthogonal polynomials on the finite interval [- 1 , + 1], with respect to the continuous weight function involving the couple of real free parameters, α and β.

Many problems in continuous or discrete domain can be solved by applying the extremal properties of Christoffel-Darboux sum.

Many actual design problems can be solved by choosing a filter function as result of the approximation technique by orthogonal polynomials known as the classical, as they are: Gegenbauer (ultraspherical), Chebyshev (first and second k ind) and Legendre (spherical) polynomials. Some of applications of orthogonal polynomials in theory of electrical filters are considered in papers [1-18]. The Jacobi orthogonal polynomials, $P_n^{(\alpha,\beta)}(x)$, hold a key

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position in a hierarchy of orthogonal polynomial classes. For certain choices of the real parameter values, α and β, several classes of orthogonal polynomials considered "classical" were produced as special or limiting cases of the Jacobi polynomials class. Therefore, finding a filter function class attributed to Jacobi orthogon al polynomials was a challenging task. In this paper proposed approximation inherit the extremal property of the parent Jacobi orthogonal polynomials and demonstrate their suitability in modern filter designing applications.

By solving it task we found that derived approximation, which proposed in this work, can be used to generate set of new filter function and set of par ticular solutions wich generate in literature well known classical filter functions.

In this paper, we propose an original two-dimensional symmetric Christoffel-Darboux formula for classical orthonormal Chebyshev polynomials of the second kind in the compact explicit form, which is valid for even and odd order. The examples are illustrated and tables of two-dimensional symmetric functions of low order are given.

2 One-dimensional classical Christoffel-Darbuova formula for ortonormal Chebyshev polynomials of the second kind

The general form of a polynomial function of n-th order is given in the following expression:

$$\phi_n(x) = \sum_{k=0}^{n} a_k x^k \tag{1}$$

Chebyshev orthogonal polynomials of n-th order of the second kind, $U_n(x)$, the real variable x, are given in the following expression:

$$U_n(x) = \frac{n}{2}\sum_{k=0}^{[n/2]}(-1)^k\frac{(n-k-1)!}{k!(n-2k)!}(2x)^{n-2k} \tag{2}$$

Chebyshev orthogonal polynomials are orthogonal on interval [-1, 1] with the weighting function $\omega(x)$

$$\omega(x) = \sqrt{1-x^2} \tag{3}$$

Where the norm is

$$h_r = \int_{-1}^{1}\sqrt{1-x^2}\,U_r(x)\,U_r(x)dx = \left\{\frac{\pi}{2},\quad r \neq 0\right. \tag{4}$$

Example 1:

For even $n, n = 12$, (i.e. $n = 2r$) Chebyshev polynomials of the second kind, $U_{12}(x)$, has the form

$$U_{12}(x) = 1 - 84x^2 + 1120x^4 - 5376x^6 + 11520x^8 - 11264x^{10} + 4096x^{12}$$

and it is shown in Fig. 1.

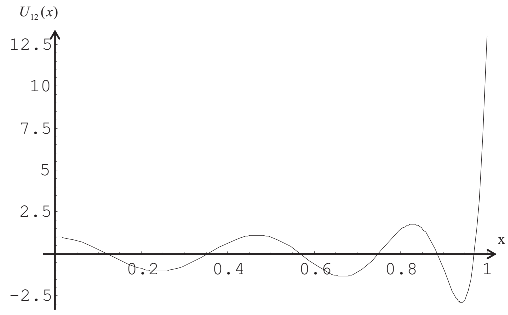

Fig. 1: Chebyshev polynomials of the second kind for even order n, n=12, in the normalized interval

Example 2:

For odd $n, n = 15$, (i.e. $n = 2r+1$) Chebyshev polynomials of the second kind, $U_{15}(x)$, has the form

$$U_{15}(x) = -16x + 672x^3 - 8064x^5 + 42240x^7 - 112640x^9 + 159744x^{11} - 114688x^{13} + 32768x^{15}$$

and it is shown in Fig. 2.

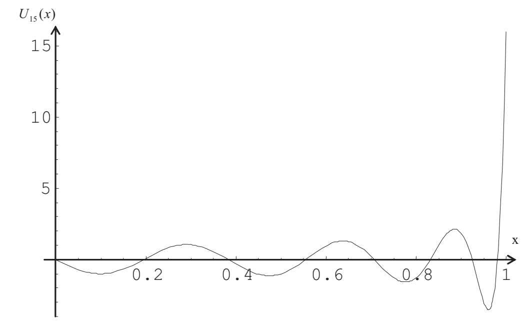

Fig. 2: Chebyshev polynomials of the second kind for odd order n, n=15, in the normalized interval

Extremal properties of the polynomial continuous classical one-dimensional real function is given by Christoffel-Darboux formula for Chebyshev orthogonal polynomials of the second kind and has the form:

$$\Phi_n(x) = \frac{[U_0(x)]^2}{\frac{\pi}{2}} + \frac{[U_1(x)]^2}{\frac{\pi}{2}} + \frac{[U_2(x)]^2}{\frac{\pi}{2}} + \ldots + \frac{[U_{n-1}(x)]^2}{\frac{\pi}{2}} + \frac{[U_n(x)]^2}{\frac{\pi}{2}} \tag{5}$$

Based on the expression (5), Table 1 shows examples of polynomial functions of Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind for different values.

Based on equality (5) fundamental observations were performed in [20], known in the literature as Bessel's and Parserval's formula.

Table 1: One-dimensional classical Christoffel-Darboux formula for the given order n

n	$\left(\frac{\pi}{2}\right)\ \Phi_n(x) = \sum_{r=o}^{n} U_r(x)\cdot U_r(x)$
0	1
1	$1+4x^2$
2	$2-4x^2+164x^4$
3	$2+12x^2-48x^4+64x^6$
4	$3-12x^2+128x^4-320x^6+256x^8$
5	$3+24x^2-256x^4+1088x^6-1792x^8+1024x610$
6	$4-24x^2+480x^4-2880x^6+7680x^8-9216x^{10}+4096x^{12}$
7	$4+40x^2-800x^4+6592x^6-25088x^8+48128x^{10}-45056x^{12}+16384x^{14}$
8	$5-40x^2+1280x^4-13504x^6+68864x^8-187392x^{10}+$ $278528x^{12}-212992x^{14}+65536x^{16}$
9	$5+60x^2-1920x^4+25536x^6-166656x^8+602112x^{10}-$ $1261568x^{12}+1523712x^{14}-983040x^{16}+262144x^{18}$
10	$6-60x^2+2800x^4-45248x^6+366592x^8-1683456x^{10}+$ $4653056x^{12}-7880704x^{14}+7995392x^{16}-4456448x^{18}+1048576x^{20}$
11	$684x^2-3920x^4+76160x^6-747520x^8+4231168x^{10}-$ $14778368x^{12}+32849920x^{14}-46530560x^{16}+40632320x^{18}-$ $19922944x^{20}+4194304x^{22}$
12	$7-84x^2+5376x^4-122752x^6+1433088x^8-9768960x^{10}+$ $41828352x^{12}-116932608x^{14}+216465408x^{16}-262930432x^{18}+$ $201326592x^{20}-88080384x^{22}+16777216x^{24}$
13	$7+112x^2-7168x^4+190848x^6-2609664x^8+21039104x^{10}-107954176x^{12}+$ $368328704x^{14}-854130688x^{16}+1351090176x^{18}-1434451968x^{20}+$ $977272832x^{22}-385875968x^{24}+67108864x^{26}$
14	$8-112x^2+9408x^4-287616x^6+4549632x^8-42747904x^{10}+$ $258236416x^{12}-1051541504x^{14}+2965110784x^{16}-585772672x^{18}+$ $8071938048x^{20}-7604273152x^{22}+4664066048x^{24}-$ $1677721600x^{26}+268435456x^{28}$

3 A novel two-dimensional continuous Christoffel-Darboux formula for Chebyshev polynomials of the second kind

The proposed two-dimensional continuous symmetric Christoffel-Darboux formula for orthonormal Chebyshev polynomials of the second kind is the generalization of the results of $\Phi_n(x,y)$, and it is given by the formula (5). The new formula $\Phi_n(x,y)$ is defined for two real variables as

$$\begin{aligned}\Phi_n(x,y) = &\frac{U_0(x)\,U_0(x)\cdot U_0(y)\,U_0(y)}{\sqrt{h_0\cdot h_0\cdot h_0\cdot h_0}} + \frac{U_1(x)\,U_1(x)\cdot U_1(y)\,U_1(y)}{\sqrt{h_1\cdot h_1\cdot h_1\cdot h_1}} + \\ &\vdots \\ &+\frac{U_{n-1}(x)\,U_{n-1}(x)\cdot U_{n-1}(y)\,U_{n-1}(y)}{\sqrt{h_{n-1}\cdot h_{n-1}\cdot h_{n-1}\cdot h_{n-1}}} + \frac{U_n(x)\,U_n(x)\cdot U_n(y)\,U_n(y)}{\sqrt{h_n\cdot h_n\cdot h_n\cdot h_n}}\end{aligned} \tag{6}$$

respectively

$$\Phi_n(x,y) = \sum_{r=1}^{n} \frac{U_r(x)\ U_r(x)\ U_r(x)\ U_r(x)}{\left(\frac{\pi}{2}\right)^2} \tag{7}$$

Homogeneous part of function, $\Phi(x) - \Phi(0)$, is of interest for solving problems in engineering, physics and atomic physics, and it is presented in the form for two real variables

$$\Phi_n(x,y) - \Phi_n(0,0) \tag{8}$$

or normalized form of the homogeneous part of function

$$\frac{\Phi_n(x,y) - \Phi_n(0,0)}{\Phi_n(1,1) - \Phi_n(0,0)} \tag{9}$$

In Table 2 a two-dimensional function Christoffel-Darboux formula for orthonormal Chebyshev polynomials of the second kind, for polynomials order $n = 1,2,\ldots,7$, is given.

For even order, $n = 12$, the Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind $\Phi_{12}(x)$is illustrated in Fig. 3.

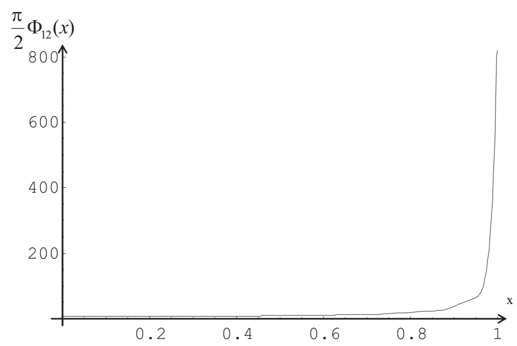

Fig. 3: For even order, $n = 12$, the Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind $\Phi_{12}(x)$

For odd order, $n = 15$, Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind $\Phi_{15}(x)$ is illustrated in Fig. 4.

Figures 5, 6 and 7 illustrate the examples of odd order $(n = 11)$ of proposed two-dimensional Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind.

Figures 8, 9 and 10 illustrate the examples of even order $(n = 16)$ proposed two-dimensional Christoffel-Darboux formula for orthonormal Chebyshev polynomials of the second kind.

Table 2: Continuous two-dimensional Christoffel-Darboux formula for orthonormal Chebyshev polynomials of the second kind, for different values of n

n	$\left(\frac{\pi}{2}\right)^2\ \Phi_n(x,y)$
1	$16y^2x^2$
2	$1-8y^2+16y^4-8x^2+80y^2x^2-128y^4x^2+16x^4-128y^2x^4+256y^4x^4$
3	$1-8y^2+16y^4-8x^2+336y^2x^2-1152y^4x^2+1024y^6x^2+16x^4-1152y^2x^4$ $+4352y^4x^4-4096y^6x^4+1024y^2x^6-4096y^4x^6+4096y^6x^6$
4	$2-32y^2+192y^4-384y^6+256y^8-32x^2+912y^2x^2-5376y^4x^2+10240y^6x^2$ $-6144y^8x^2+192x^4-5376y^2x^4+35328y^4x^4-71680y^6x^4+45056y^8x^4-384x^6$ $+10240y^2x^6-71680y^4x^6+151552y^6x^6-98304y^8x^6+256x^8-6144y^2x^8$ $+45056y^4x^8-98304y^6x^8+65536y^8x^8$
5	$2-32y^2+192y^4-384y^6+256y^8-32x^2+2208y^2x^2-19200y^4x^2+60928y^6x^2$ $-79872y^8x^2+36864y^{10}x^2+192x^4-19200y^2x^4+182784y^4x^4-612352y^6x^4$ $+831488y^8x^4-393216y^{10}x^4-384x^6+60928y^2x^6-612352y^4x^6+2134016y^6x^6$ $-2981888y^8x^6+1441792y^{10}x^6+256x^8-79872y^2x^8+831488y^4x^8$ $-2981888y^6x^8+4259840y^8x^8-2097152y^{10}x^8+36864y^2x^{10}-393216y^4x^{10}$ $+1441792y^6x^{10}-2097152y^8x^{10}+1048576y^{10}x^{10}$
6	$3-80y^2+928y^4-4352y^6+9728y^8-10240y^{10}+4096y^{12}-80x^2$ $+4512y^2x^2-54528y^4x^2+251392y^6x^2-534528y^8x^2+528384y^{10}x^2-196608y^{12}x^2$ $+928x^4-54528y^2x^4+724480y^4x^4-3532800y^6x^4+7802880y^8x^4-7929856y^{10}x^4$ $+3014656y^{12}x^4-4352x^6+251392y^2x^6-3532800y^4x^6+17879040y^6x^6$ $-40566784y^8x^6+42074112y^{10}x^6-16252928y^{12}x^6+9728x^8-534528y^2x^8$ $+7802880y^4x^8-40566784y^6x^8+93978624y^8x^8-99090432y^{10}x^8$ $+38797312y^{12}x^8-10240x^{10}+528384y^2x^{10}-7929856y^4x^{10}+42074112y^6x^{10}$ $-99090432y^8x^{10}+105906176y^{10}x^{10}-41943040y^{12}x^{10}+4096x^{12}-196608y^2x^{12}$ $+3014656y^4x^{12}-16252928y^6x^{12}+38797312y^8x^{12}-41943040y^{10}x^{12}+16777216y^{12}x^{12}$
7	$3-80y^2+928y^4-4352y^6+9728y^8-10240y^{10}+4096y^{12}-80x^2+8608y^2x^2$ $-136448y^4x^2+857600y^6x^2-2631680y^8x^2+4198400y^{10}x^2$ $-3342336y^{12}x^2+1048576y^{14}x^2+928x^4-136448y^2x^4+2362880y^4x^4$ $-15656960y^6x^4+49745920y^8x^4-81330176y^{10}x^4+65929216y^{12}x^4$ $-20971520y^{14}x^4-4352x^6+857600y^2x^6-15656960y^4x^6$ $+107597824y^6x^6-350945280y^8x^6+585236480y^{10}x^6-481820672y^{12}x^6$ $+155189248y^{14}x^6+9728x^8-2631680y^2x^8+49745920y^4x^8$ $-350945280y^6x^8+1167720448y^8x^8-1978138624y^{10}x^8+1649410048y^{12}x^8$ $-536870912y^{14}x^8-10240x^{10}+4198400y^2x^{10}-81330176y^4x^{10}$ $+585036480y^6x^{10}-1978138624y^8x^{10}+3394240512y^{10}x^{10}-2860515328y^{12}x^{10}$ $+939524096y^{14}x^{10}+4096x^{12}-3342336y^2x^{12}+65929216y^4x^{12}$ $-481820672y^6x^{12}+1649410048y^8x^{12}-2860515328y^{10}x^{12}+2432696320y^{12}x^{12}$ $-805306368y^{14}x^{12}+1048576y^2x^{14}-20971520y^4x^{14}+155189248y^6x^{14}$ $-536870912y^8x^{14}+939524096y^{10}x^{14}-805306368y^{12}x^{14}+268435456y^{14}x^{14}$

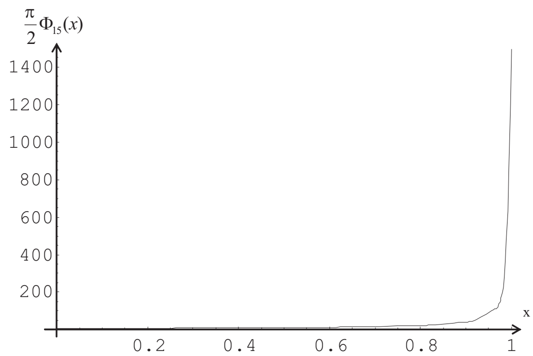

Fig. 4: For odd order, $n = 15$, the Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind $\Phi_{15}(x)$

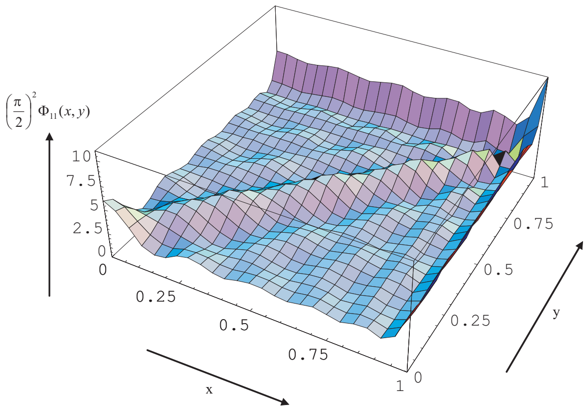

Fig. 5: 3D plot of the proposed symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind odd order, n=11

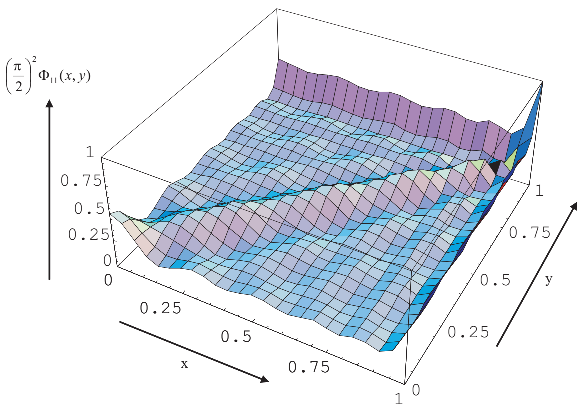

Fig. 6: 3D plot of the proposed normalized symetric two-dimensinal continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind odd order, n=11

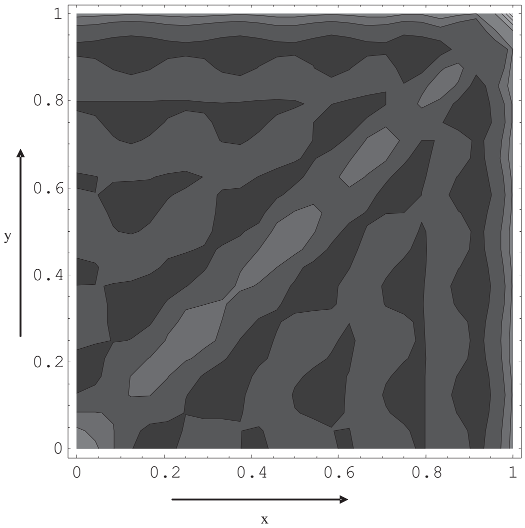

Fig. 7: 2D contour plot of the proposed symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind odd order, n=11

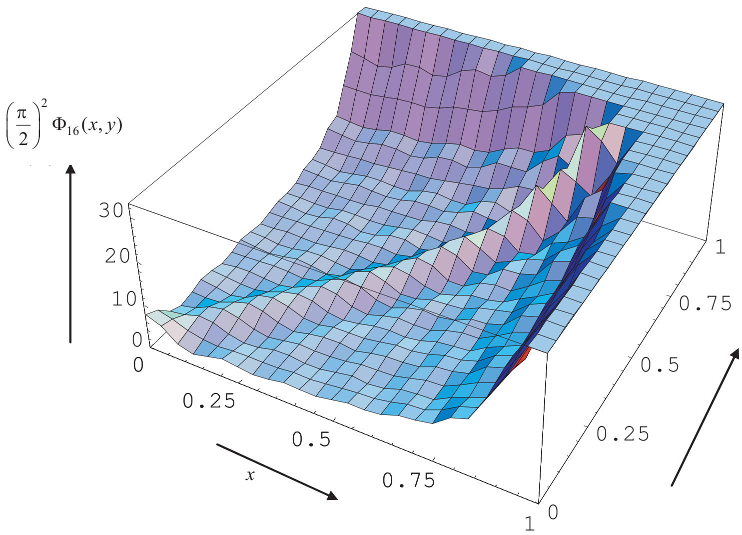

Fig. 8: 3D plot of the proposed symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind even order, n=16

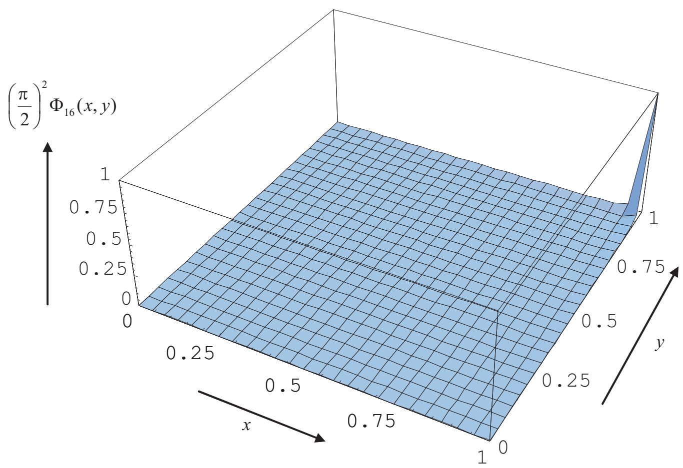

Fig. 9: 3D plot of the proposed normalized symetric two-dimensinal continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind even order, n=16.

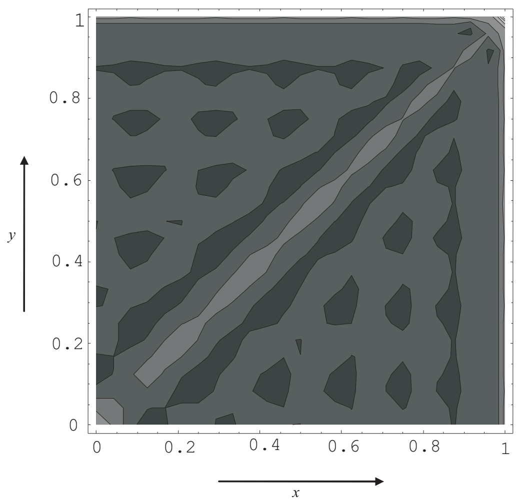

Fig. 10: 2D contour plot of the proposed symmetric two-dimensional continuous Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind even order, n=16.

4 Conclusion

Symmetric two-dimensional continual Christoffel-Darboux formula for orthogonal Chebyshev polynomials of the second kind in compact explicit form is presented in this paper. The formula is general and applies to the even and odd order for two real variables.

The examples of even and odd order of the proposed formula are illustrated and shown in the table of two-dimensional real function, $\Phi_n(x,y)$, which is generated by the proposed formula, and is applicable in the technique for the synthesis of analog and digital filter functions.

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References

[1] B.D. Raković, *Designing monotonic low-pass filters-comparison of some me thods and criteria*, Circuit Theory and Applications, Vol. 2, 215-221, 1974.

[2] D. Johnson, J. Johnson, *Low-pass filters using ultraspherical polynomials , IEEE Trans. Circuit Theory*, CT-13, 364-369, 1966.

[3] C. Beccari, *The use of the shifted Jacobi polynomials in the synthesis of lowpass filters* , Circuit Theory and Applications, Vol. 7, 289-295, 1979.

[4] B.D. Raković, M. V. Popović, *Explicit expression for the characteristic function of generalized Legendre filters*, Circuit Theory and Applications, Vol. 6, 363-373, 1978.

[5] B. D. Raković, *Characteristic functions for least mean square approximation for all pole filters* , Publ. of Electrical Engineering Faculty, University of Belgrade, 107-108,(1972) 23-26.

[6] B. D. Rakovich, *Predistortion techniques for increasing the element tolerances in equiriple passband filters revised*, Int. J. Electronics, Vol. 54, No. 6, pp. 905-912, 1983.

[7] B. D. Rakovich, *Transitional Butterworth-Legendre Filters*, Radio& Electron. Eng., 44, pp. 673-680, 1974.

[8] V. D. Pavlović, *Least-Square Low-pass Filters Using Chebyshev Polynomials*, Int. J. Electronics, Vol. 53, No. 4, pp. 371-379, UK, 1982.

[9] V. D. Pavlović, *Direct Synthesis of Filter Transfer Functions*, IEE Proceedings, Vol. 131, Pt. G. No. 4, pp. 156-160, UK, Avg. 1984.

[10] V. D. Pavlović, *Filter Transfer Function Synthesis by Hermite Generating Function*, Journal of Electrotechn. Math., Vol. 9, No. 1, pp. 35-41, Kosovska Mitrovica, 2004.

[11] V. D. Pavlović, *Synthesis of Filter Function Using Generating Functions of Classical Orthogonal Polynomials*, Journal of Technical Sciences and Mathematics, Vol. 10, No. 1, pp. 65-76, Kosovska Mitrovica, 2005.

[12] M. Lutovac, D. Rabrenović, *All-pole filters using ultraspherical polynomials*, European Conf. Circuit Theory Design, ECCTD'91, pp.203-212, Copenhagen, Sep. 1991.

[13] V. D. Pavlović, *Filter transfer function synthesis by Gegenbauer generating function*, YU Proceedings of the XXVII Conference of ETAN, Part III, pp. 157-164, Sarajevo, 6-10. June 1988.

[14] G. V. MILOVANOVIĆ, V. D. PAVLOVIĆ, *Uslovna srednje-kvadratna aproksimacija prenosne funkcije sa čebiševljevom težinom* , Numeričke metode u tehnici (III Znanstveni skup), Stubike toplice, 1981, pp. 241-247.

[15] A. I. ZVEREV, *Handbook of Filter Synthesis*, John Wiley, New York, 1976.

[16] R. SALL, W. ENTENMANN, *Handbook of Filter Design*, AEG-TELEFUNKEN, Berlin, 1979.

[17] B. D. RAKOVIĆ, V. D. PAVLOVIĆ, *Method of designing doubly terminated lossy ladder filters with increased element tolerances*, IEE Proceedings, Vol. 134, Pt. G. No. 6, pp. 285-291, UK, Dec. 1987.

[18] V. D. PAVLOVIĆ, M. V. POPOVIĆ, *An Iterative Method for Lossy LC Ladder filter synthesis*, European Conference on Circuit Theory and design Proceedings, Paris, pp. 185-190. Sep. 1987.

[19] G. SZEGO, *Orthogonal Polynomials, American Mathematical Society*, Colloquium Publications, XXIII, New York, USA, 1939.

[20] A. ANGOT, *Complements de mathematiques*, A lusage des ingenieurs de l'Elektrotechnique et des telecomunicationss, third ed., Paris, 1957, (Russian translatiom: Nauka, Moscow, 1976).

[21] M. ABRAMOWITZ, I. A. STEGUN, *Handbook of Mathematical Functions*, Dover Publications, Inc., New York, USA, 1964.

[22] R. WESLES, *Numerical methods for scientists and engineers*, Bell telephone laboratories, McGraw- Hill, New York, USA, 1962.

[23] L. C. ANDREWS, *Special Functions of Mathematics for Engineers*, 2nd ed., Oxford., U.K., Oxford University Press, 1998.

[24] D. MITRINOVIĆ, *Uvod u specijalne funkcije*, Gradjevinska knjiga, Beograd, 1972.

[25] D. MITRINOVIĆ, D. DJOKOVIĆ, *Special function*, Gradjevinska knjiga, Beograd, 1964.

[26] S. GHOH, *Generalized Christoffel-Darboux formula for skew-orthogonal polynomials and random matrix theory*, JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL, VOL. 39, NO. 28, PP. 8775-8782, DOI: 10.1088/0305-4470/39/28/S02, 2006.

[27] S. GHOSH, *Generalized Christoffel-Darboux formula for classical skew-orthogonal polynomials*, JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL, VOL. 41, NO. 43, P 435204, DOI:10.1088/1751-8113/41/43/435204, 2008.

[28] A. LASCOUX, P. PRAGACZ, BEZOUTIANS, *Euclidean Algorithm and Orthogonal Polynomials*, ANNALS OF COMBINATORICS, VOL. 9, ISSUE 3, P 301, 2005.

[29] P. W. AYERS, *Generalized Christoffel-Darboux formulae and the frontier Kohn-Sham molecular orbitals*, THEORETICAL CHEMISTRY ACCOUNTS: THEORY, COMPUTATION AND MODELING, VOL. 110, ISSUE 4, PP. 267-275, DOI: 10.1007/S00214-003-0446-2, 2003.

[30] M. A. JAFARIZADEH, R. SUFIANI, S. JAFARIZADEH, *Recursive calculation of effective resistances in distance-regular networks based on Bose-Mesner algebra and Christoffel-Darboux identity*, JOURNAL OF MATHEMATICAL PHYSICS, VOL. 50, ISSUE 2, 023302, 2009, HTTP://DX.DOI.ORG/10.1063/1.3077145.

[31] S. YING, *On generalized Christoffel functions*, ACTA MATHEMATICA HUNGARICA, Vol. 135, Issue 3, p 213, May 2012.