

## Finite Systems of Equations and Implicit Functions

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**Abstract:** The article deals with the well known Rückert – Lefschetz scheme of investigation of implicit functions that are defined with finite systems  $f(x) = 0$  of equations with analytical left hand sides. It is proved that this scheme is not effective. This means that this scheme does not allow to define jets of implicit functions using only jets of left hand sides of equations under considerations even for structurally stable systems of equations (although it allows to describe the possible structure of the set of implicit functions). There is presented some modification of the Rückert-Lefshetz scheme which, in basic cases, allows to define jets of implicit functions using only jets of left hand sides of equations under considerations even for structurally stable systems of equations.

**Keywords:** Rückert – Lefschetz scheme, implicit function, effectively computable solution

### 1 Introduction

Let us consider a finite system of the equations

$$\begin{cases} f_1(\lambda, x_1, \dots, x_m) = 0, \\ \dots\dots\dots\dots\dots\dots\dots \\ f_n(\lambda, x_1, \dots, x_m) = 0, \end{cases} \quad (1)$$

where the parameter  $\lambda$  and the unknowns  $x_1, \dots, x_m$  are the real or complex numbers and  $f_j(\lambda, x_1, \dots, x_m)$  ( $j = 1, \dots, n$ ) are the real or complex valued functions. System (1) can be shown as

$$f(\lambda, x) = 0, \quad (2)$$

where  $f(\cdot, \cdot)$  is a map of  $\mathbb{R} \times \mathbb{R}^m$  to  $\mathbb{R}^m$  or  $\mathbb{C} \times \mathbb{C}^m$  to  $\mathbb{C}^m$ .

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Suppose

$$f(\lambda_0, x_0) = 0.$$

In different analysis problems (the differential and integral equations, the methods of optimization and etc) it often appears the following question: when does System (2) define in a local neighborhood of the point  $(\lambda_0, x_0)$  (or in some part of this neighborhood) one or several continuous functions  $x(\lambda)$  at the point  $\lambda_0$  such that  $x(\lambda_0) = x_0$ ? These functions are often called *implicit functions* or *small solutions* of System (1).

The classical theorem about the implicit functions is well-known [4, 10]: *if  $m = n$ ,  $f(\lambda, x)$  is a continuous function at the point  $\lambda_0$ ,  $f'_x(\lambda, x)$  is a continuous function at the point  $(\lambda_0, x_0)$  and  $f'_x(\lambda_0, x_0)^{-1}$  also exists, then System (2) has the unique solution  $x = x^*(\lambda)$  in a small neighborhood of the point  $(\lambda_0, x_0)$ . This case is called *nondegenerated*. If  $f'_x(\lambda_0, x_0)$  is an irreversible matrix, then the such case is called *degenerated*.*

The analysis of the degenerated cases is a difficult problem. The basic results concern to the case when  $f_j(\lambda, x_1, \dots, x_m)$  ( $j = 1, \dots, n$ ) are analytical functions in a neighborhood of the point  $(\lambda_0, x_0)$  (see, for example, [8, 2, 4]); some of these results are extended to the case when the functions  $f_j(\lambda, x_1, \dots, x_m)$  are smooth enough.

Depending on what of the cases  $m = n$ ,  $m > n$ ,  $m < n$  takes place, it is said that System (1) is *determined*, *underdetermined* and *overdetermined*. It seems that the determined systems should define the finite number of the solutions  $x(\lambda)$ , underdetermined ones should define the infinite number of such solutions and the overdetermined ones should not define the such solutions in general. However the distinction between these three types of the systems is conditional. So if we add one or several equation so that the number of equations became the same with the number of unknowns then the underdetermined system became determined. The overdetermined systems also can be considered as the determined systems if the left parts of its equations depend on also  $n - m$  unknowns  $x_{m+1}, \dots, x_n$ .

In the article (if it is not stipulated the opposite) the case when the parameter  $\lambda$  and the unknowns  $x_1, \dots, x_m$  take the complex values is considered. There are situations when the founded solutions  $\mathfrak{L}$  at the point  $\lambda_0$ . To avoid the consideration of the multiple-valued functions in the such cases it is natural to consider the implicit functions defined by the system (1) in the neighborhood of the point  $\lambda_0$  with a cross-cut. The case when the parameter  $\lambda$  and unknowns  $x_1, \dots, x_m$  take real values will be in details considered in the second part of this article.

Assume that the functions  $f_j(\lambda, x_1, \dots, x_m)$  ( $j = 1, \dots, n$ ) are analytical. Then the zero set  $\mathfrak{N} = \{(\lambda, x) \in \mathbb{C}^{m+1} : \lambda \neq \lambda_0, f(\lambda, x) = 0\}$  of the left parts of System (1) in the neighborhood of the point  $(\lambda_0, x_0)$  can be presented in the form  $\mathfrak{N} = \mathfrak{N}_0 \cup \mathfrak{N}_1 \dots \cup \mathfrak{N}_m$ . Here the set  $\mathfrak{N}_0$  is empty or consists of a finite number of the graphs of solutions  $x = \phi(\lambda)$  where  $\phi(\lambda)$  are some analytical functions of the parameter  $(\lambda - \lambda_0)^{\frac{1}{r}}$  ( $r$  is a natural number). Further, each of the sets  $\mathfrak{N}_j$

( $j = 1, \dots, m - 1$ ) is empty or consists of a finite number of the surfaces  $\mathfrak{L}_j$  that are graphs of functions of type  $x = \phi(\lambda; \xi_1, \dots, \xi_j)$  where  $\xi_s$  ( $s = 1, \dots, j$ ) are free parameters (Fig. 1). Moreover, the functions  $\phi(\lambda; \xi_1, \dots, \xi_j)$  ( $j = 1, \dots, m - 1$ ) have

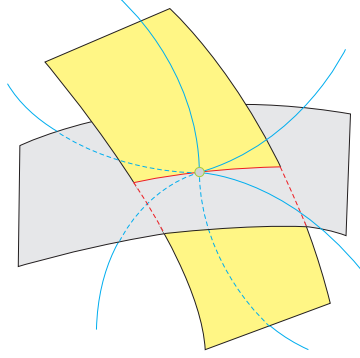


Fig. 1. The zero set  $\mathfrak{N}$

the following property: if we replace in these functions the parameters  $\xi_1, \dots, \xi_j$  ( $s = 1, \dots, j$ ) by some analytical functions  $\xi_s(\lambda)$  ( $s = 1, \dots, j$ ) depending on parameters  $(\lambda - \lambda_0)^{\frac{1}{p}}$  ( $p$  is a natural) then the superpositions  $\phi(\lambda; \xi_1(\lambda), \dots, \xi_j(\lambda))$  also will be analytical functions of the parameter  $(\lambda - \lambda_0)^{\frac{1}{q}}$  ( $q$  is also a natural, and  $p$  is a divisor of  $q$ ). At last, the set  $\mathfrak{N}_m$  is not empty only when System (1) is trivial, i. e. when its left parts are identical to the zero (in this case any continuous in a point  $\lambda_0$  function  $x(\lambda)$  such that  $x(\lambda_0) = x_0$ ) is satisfy to System (1).

As a result we can give the description of the general structure of implicit functions defined by System (1). It is obviously that the function  $x = x(\lambda)$  defined in the neighborhood of the point  $\lambda_0$  is implicit if and only if its graph lies in the set  $\mathfrak{N}$ . In particular, if the set  $\mathfrak{N}_0$  is not empty then it defines a finite set  $\Phi_0$  of the implicit functions. Each of these functions is an analytical function of the parameter  $(\lambda - \lambda_0)^{\frac{1}{r}}$  ( $r$  is a natural). Further, the sets  $\Phi_j$  ( $j = 1, \dots, m - 1$ ) of implicit functions, that are analytical functions of the parameter  $(\lambda - \lambda_0)^{\frac{1}{r}}$  ( $r$  is a natural), whose graphs lie on the surfaces  $\mathfrak{L}_j$  consisting of the set  $\mathfrak{N}_j$  ( $j = 1, \dots, m - 1$ ), are infinite provided that they are nonempty. Remark else the following: if the set  $\mathfrak{N}_j$  ( $j = 1, \dots, m - 1$ ) is not empty then there are others (continuous in a neighborhood  $\lambda_0$ !) implicit functions which are not analytical of the parameter  $(\lambda - \lambda_0)^{\frac{1}{r}}$  ( $r$  is a natural). However such functions can be excluded from the consideration as soon as the graphs of the analytical implicit functions of the set  $\Phi_j$  fill by the surfaces of the set  $\mathfrak{N}_j$  the neighborhoods of the point  $(\lambda_0, x_0)$ .

Let us denote by  $\Phi$  the set of all analytical implicit functions  $x = x(\lambda)$  of the parameters  $(\lambda - \lambda_0)^{\frac{1}{r}}$  ( $r$  is a natural) turning into  $x_0$  at  $\lambda = \lambda_0$ . Obviously  $\Phi = \Phi_0 \cup \Phi_1 \cup \dots \cup \Phi_m$ .

The formulated statements was proved at the first half of XX-th century, as it is known to the authors, by W. Rückert [7]. The more modern statement of these

results can be found in the monographs [1, 3] (see also, [4]). The corresponding arguments were based on Kronecker elimination theory for the systems of the algebraic equations and on Weierstrass preparation theorem for analytical functions of the complex variable. In the monograph of S. Lefschetz [6] (he investigated the special systems of the type (1) which arose in the problem about the periodic solutions of the ordinary differential equations) the more elementary statement of the results about the structure of the set of the implicit functions was given.

Though the S. Lefschetz arguments were not constructive, they laid down in a basis of the general constructions of M. M. Vainberg and V. A. Trenogin. In the monograph [8] they stated that their scheme allows them to give the complete description of the sets  $\Phi_0, \Phi_1, \dots, \Phi_m$  and, moreover, to define the first coefficients in the expansions of the series along the parameter  $\lambda$  or at its fractional degrees of the solutions from set  $\Phi_0$ . In the monograph [4] was noticed that it is not truth. In this monograph it was shown that to define the first coefficients of the expansions of the series of the solutions of the general system (1) probably only for so-called the simple solutions (the solution  $\phi(\lambda)$  of the systems (1) is *simple* if  $m = n$  and for close to  $\lambda_0$  and distinct from  $\lambda_0$  values  $\lambda$  the Jacobian  $\det f'_x(\lambda, \phi(\lambda))$  is non zero). Moreover, in this monograph it was shown that scheme of M. M. Vainberg and V. A. Trenogin does not allow (if we use in the calculations only the finite numbers coefficients of the expansions of the left part of System (1) at the series) to define the number of the implicit functions of the set  $\Phi_0$  and the coefficients of the first members of the expansions of these implicit functions even when  $m = n = 3$  (and in essence when  $m = n = 2$ ). In this monograph also was shown the special example of the system (when  $m = n = 3$ ) when some updating of the Lefschetz scheme allow to define the structure of the set  $\Phi_0$ .

In the following section the Rückert–Lefschetz scheme will be analyzed in details. Besides in this section we emphasize moments which make the Rückert–Lefschetz scheme not constructive and, moreover, the Rückert–Lefschetz scheme does not allow to define the structure of the set of the implicit functions defined by System (1) even for the rough systems. (In this article System (1) is called *rough* if it has only the finite number of the simple solutions (in [4] the term *rough systems* was used in a bit different sense). It is known that System (1) is rough if and only if it possesses to the following *stability property*: for every big enough natural  $N$  there exists a natural  $\tilde{N}$  such that if we change the members in the left parts of System (1) whose orders are higher than  $\tilde{N}$  then the number of the solutions of System (1) does not change and, moreover, the first  $N$  members of the expansions in the series of these solutions also do not change.) In the fourth section some modified scheme of the research of System (1) is offered; the basic idea of this modification is due to the mentioned above example from [4].

Let us notice that the Rückert–Lefschetz scheme is not unique. The various statements about the structure of the implicit functions defined by the system (1) have been received by V. V. Pokornyi, P. P. Rybin, V. B. Melamed, A. E. Gel'man;

the considerable part of the results of these authors is summarized in the monograph [4]. It is necessary to note the work [2] of N. P. Erugin separately because his work contain a number of theorems about implicit functions which based on the construction of the jets (the sum of first members in expansions in power series) of the expansions of these functions at the series.

The proof of the Rückert–Lefschetz scheme use only the elementary means of algebra and the theory of functions of complex variables (i. e., such classical concepts as the resultant, the greatest common divisor of the polynomials with the coefficients from the factorial rings (i. e. the rings with the unique factorization on primes), etc.). The abstract theory of the polynomial ideals is not used. In this article we use the results on the theory of implicit functions which described in [4].

## 2 The Rückert–Lefschetz scheme

Below we assume that  $\lambda_0 = 0$ ,  $x_0 = 0$  and  $f_j(\lambda, x_1, \dots, x_m) \not\equiv 0$  ( $j = 1, \dots, n$ ). From the last assumption follows that  $\Phi_m = \emptyset$ .

We change the designations of functions  $f_j(\lambda, x_1, \dots, x_m)$  onto  $f_j^{(m)}(\lambda, x_1, \dots, x_m)$  ( $j = 1, \dots, n$ ) (in what follows, it is convenient to fix the number of unknowns in designations). Since the functions  $f_j^{(m)}(\lambda, x_1, \dots, x_m)$ ,  $j = 1, \dots, n$ , are analytical, we present the functions  $f_j^{(m)}(\lambda, x_1, \dots, x_m)$  ( $j = 1, \dots, n$ ) in the form of the converging series in some neighborhood of the zero

$$f_j^{(m)}(\lambda, x_1, \dots, x_m) = \sum_{k_0+k_1+\dots+k_m=1}^{\infty} a_{k_0, k_1, \dots, k_m} \lambda^{k_0} x_1^{k_1} \dots x_m^{k_m} \quad (j = 1, \dots, n). \quad (3)$$

We divide each equation of System (3) on the highest possible degree of  $\lambda$  and so, without the loss of generality, we can assume

$$f_j^{(m)}(0, x_1, \dots, x_m) \not\equiv 0 \quad (j = 1, \dots, n).$$

In addition we make a linear substitution of the unknowns  $x_1, \dots, x_m$  so that the functions

$$f_j^{(m)}(0, 0, \dots, 0, x_m) \quad (j = 1, \dots, n) \quad (4)$$

turn out to be nonzero.

As a result of the application of the Weierstrass preparation theorem [3] to each function  $f_j^{(m)}(\lambda, x_1, \dots, x_m)$  ( $j = 1, \dots, n$ ) we receive the equalities

$$f_j^{(m)}(\lambda, x_1, \dots, x_m) = \varepsilon_j^{(m)}(\lambda, x_1, \dots, x_m) \cdot \tilde{f}_j^{(m)}(\lambda, x_1, \dots, x_m), \quad (5)$$

where  $\varepsilon_j^{(m)}(\cdot)$  is an analytical function at the zero, such that  $\varepsilon_j^{(m)}(0) \neq 0$ ;  $\tilde{f}_j^{(m)}(\lambda, x_1, \dots, x_m)$  is a polynomial of the unknown  $x_m$  whose coefficients are analytical at the zero functions of the parameter  $\lambda$  and unknowns  $x_1, \dots, x_{m-1}$ .

From the equalities (5) follows that the search of implicit functions defined by System (1) is equivalent to the analysis of the system of algebraic equations with respect to the unknown  $x_m$ :

$$\tilde{f}_j^{(m)}(\lambda, x_1, \dots, x_m) = 0 \quad (j = 1, \dots, n). \quad (6)$$

Notice that the superior coefficients of the polynomials of  $x_m$  in the left parts of System (6) are equal 1 and all other coefficients of these polynomials are analytical functions of the parameter  $\lambda$  and unknowns  $x_1, \dots, x_{m-1}$  turning into the zero at the zero. This follows from our construction.

Let us remind (see, for example, [4, 6, 9]) that the rings of analytical at the zero functions of a finite number of variables are factorial (a ring is called factorial if it has identity element, has no divisors of the zero, and its elements are (uniquely up to the order of multipliers) displayed as the product of prime multipliers). In such rings the concept of the greatest common divisor is defined and all main statements of the divisibility theory are true. In particular, the ring of polynomials with coefficients from a factorial ring itself is a factorial ring.

Let us denote by  $d^{(m)}(\lambda, x_1, \dots, x_m)$  the greatest common divisor of polynomials  $\tilde{f}_j^{(m)}(\lambda, x_1, \dots, x_m)$ ,  $j = 1, \dots, n$ . Then

$$\tilde{f}_j^{(m)}(\lambda, x_1, \dots, x_m) = d^{(m)}(\lambda, x_1, \dots, x_m) \cdot \tilde{f}_j^{(m)}(\lambda, x_1, \dots, x_m) \quad (j = 1, \dots, n). \quad (7)$$

Hence System (1) is equivalent to the collection consisting of one algebraic equation with the unknown  $x_m$

$$d^{(m)}(\lambda, x_1, \dots, x_m) = 0, \quad (8)$$

and the system of the algebraic equations with the unknown  $x_m$

$$\tilde{f}_j^{(m)}(\lambda, x_1, \dots, x_m) = 0 \quad (j = 1, \dots, n). \quad (9)$$

If  $s$  the degree of the greatest common divisor  $d^{(m)}(\lambda, x_1, \dots, x_m)$  is positive then Equation (8), for any small enough  $\lambda, x_1, \dots, x_{m-1}$ , has  $s$  small solutions  $x_m$ . These solutions can be presented as the equations  $x_m = \phi(\lambda, x_1, \dots, x_{m-1})$ . More precisely, each of such equations defines an element of the set  $\Phi_{m-1}$  and the solutions  $x(\lambda) = (x_1(\lambda), \dots, x_{m-1}(\lambda), x_m(\lambda))$  of System (1) where  $x_i(\lambda)$  ( $i = 1, \dots, m-1$ ) are arbitrary analytical functions of  $\lambda$  or some fractional degree  $\lambda^{\frac{1}{r}}$  of  $\lambda$  turning into the zero in the zero and a component  $x_m(\lambda)$  is defined with the equation

$$x_m(\lambda) = \phi(\lambda; x_1(\lambda), \dots, x_{m-1}(\lambda))$$

where  $\phi(\lambda; x_1, \dots, x_{m-1})$  is a solution to Equation (8).

If  $s = 0$  then the greatest common divisor  $d^{(m)}(\lambda, x_1, \dots, x_m)$  does not generate the solutions of the System (6) from the set  $\Phi_{m-1}$ , and, hence, solutions of System (1).

If at least one of the functions  $\tilde{f}_j^{(m)}(\lambda, x_1, \dots, x_m)$  ( $j = 1, \dots, n$ ) is distinct from the zero at the zero point (it probably only in that case when the degree of the polynomial  $d^{(m)}(\lambda, x_1, \dots, x_m)$  coincides with the degree of one of the polynomials  $\tilde{f}_j^{(m)}(\lambda, x_1, \dots, x_m)$  ( $j = 1, \dots, n$ )) then the process of the construction of the set  $\Phi$  comes to the end. Thus  $\Phi$  coincides with  $\Phi_{m-1}$ , and  $\Phi_j$  ( $j = 0, \dots, m-2$ ) will appear the empty sets. Otherwise (i.e., when  $s$  the degree of greatest general divisor of the polynomials  $\tilde{f}_j^{(m)}(\lambda, x_1, \dots, x_m)$  ( $j = 1, \dots, n$ ) is strict less the degrees of each of these polynomials) we pass to the consideration System (9).

Let us consider the system of the equations

$$f_j^{(m-1)}(\lambda, x_1, \dots, x_{m-1}) = 0 \quad (j = 1, \dots, n_{m-1}), \quad (10)$$

whose left parts are the full system of the resultants (see, for example, [1, 4, 5]) for the polynomials standing at the left parts of System (9).

System (10) is like to the initial system (1), however, its left hand sides depend on the smaller number of the variables (namely from  $\lambda, x_1, \dots, x_{m-1}$ ). Thus if  $n = 2$ ,  $n_{m-1} = 1$  and if  $n > 2$ , the number  $n_{m-1}$  is greater the number  $n$ . The following simple statement (see, for example, [4]) will be use below.

**Lemma 1** *System of the equations (9) has the small solutions if and only if System of the equations (10) has the small solutions. More precisely, if  $x(\lambda) = (\xi_1(\lambda), \dots, \xi_{m-1}(\lambda), \xi_m(\lambda))$  is a small solution of System (9), then  $\tilde{x}(\lambda) = (\xi_1(\lambda), \dots, \xi_{m-1}(\lambda))$  is a small solution of System (10). Vice versa, if  $\tilde{x}(\lambda) = (\xi_1(\lambda), \dots, \xi_{m-1}(\lambda))$  is a small solution of System (10), then there is a finite (not equal to the zero) number of the continuous at the zero and turning into the zero in the zero of functions  $\xi_m(\lambda)$  for which  $x(\lambda) = (\xi_1(\lambda), \dots, \xi_{m-1}(\lambda), \xi_m(\lambda))$  is a small solution of System (9).*

It is possible to apply all the same arguments to the system (10) as was applied to the system (1). So having reduced the left parts of the system (10) on the possible greatest degrees of the parameter  $\lambda$  then having complete the suitable linear substitution of the unknowns and having applied the Weierstrass preparation theorem, we will receive that System (10) is equivalent to the system of the algebraic equations with respect to the unknown  $x_{m-1}$ :

$$\tilde{f}_j^{(m-1)}(\lambda, x_1, \dots, x_{m-1}) = 0 \quad (j = 1, \dots, n_{m-1}). \quad (11)$$

Let us denote by  $d^{(m-1)}(\lambda, x_1, \dots, x_{m-1})$  the greatest common divisor of the

polynomials  $\tilde{f}_j^{(m-1)}(\lambda, x_1, \dots, x_{m-1})$  ( $j = 1, \dots, n_{m-1}$ ). Then

$$\tilde{f}_j^{(m-1)}(\lambda, x_1, \dots, x_{m-1}) = d^{(m-1)}(\lambda, x_1, \dots, x_{m-1}) \cdot \tilde{f}_j^{(m-1)}(\lambda, x_1, \dots, x_{m-1})$$

$$(j = 1, \dots, n_{m-1}). \quad (12)$$

and the system of the equations (10) is equivalent to the set of one algebraic equation of the unknown  $x_{m-1}$

$$d^{(m-1)}(\lambda, x_1, \dots, x_{m-1}) = 0, \quad (13)$$

and the system of algebraic equations of the unknown  $x_{m-1}$

$$\tilde{f}_j^{(m-1)}(\lambda, x_1, \dots, x_{m-1}) = 0 \quad (j = 1, \dots, n_{m-1}). \quad (14)$$

If  $s_{m-1}$  the degree of the greatest common divisor  $d^{(m-1)}(\lambda, x_1, \dots, x_{m-1})$  is positive then the equation (13) at any enough small  $\lambda, x_1, \dots, x_{m-2}$  has  $s_{m-1}$  small solutions  $x_{m-1} = \xi(\lambda, x_1, \dots, x_{m-2})$ . Thus these solutions will define the elements of the set  $\Phi_{m-2}$ , that is the solutions of System (10)

$$x(\lambda) = \phi(\lambda; x_1(\lambda), \dots, x_{m-2}(\lambda)),$$

where  $x_i(\lambda)$  ( $i = 1, \dots, m-2$ ) are any free parameters. If  $s_{m-1} = 0$  then the greatest common divisor  $d^{(m-1)}(\lambda, x_1, \dots, x_{m-1})$  does not generate solutions of System (11), hence, and the solutions of System (10).

If at least one of the functions  $\tilde{f}_j^{(m-1)}(\lambda, x_1, \dots, x_{m-1})$  ( $j = 1, \dots, n_{m-1}$ ) is distinct from the zero at the zero point then the system of the equation (14) does not have the small solutions. Using the solutions of the equations (13) and also using Lemma 1 the solutions of the system (1) are constructed. The set of these solutions forms the set  $\Phi_{m-2}$ . Thus  $\Phi = \Phi_{m-2} \cup \Phi_{m-1}$  and the process of the construction of the set  $\Phi$  comes to the end. Otherwise we pass to the consideration of System (14).

If we apply to System (14) all the argumentation which was applied to System (9) we construct the set  $\Phi_{m-3}$ . Other sets  $\Phi_k$  ( $k = 0, \dots, m-4$ ) are constructed similarly. After the reduction on a multiplier  $\lambda$  in the suitable degrees by means of the linear change of variables and applying the Weierstrass preparation theorem to every system

$$f_j^{(k)}(\lambda, x_1, \dots, x_k) = 0 \quad (j = 1, \dots, n_k) \quad (15)$$

we obtain the system

$$\tilde{f}_j^{(k)}(\lambda, x_1, \dots, x_k) = 0 \quad (j = 1, \dots, n_k); \quad (16)$$

of the algebraic equations of the unknown  $x_k$ . Then the greatest common divisor  $d^{(k)}(\lambda, x_1, \dots, x_k)$  of the polynomials, that stands in the left parts of equations of this system is defined. At last, we construct the system

$$\tilde{f}_j^{(k)}(\lambda, x_1, \dots, x_k) = 0 \quad (j = 1, \dots, n_k). \quad (17)$$



In this moment the set  $\Phi_{k-1}$  is nonempty if and only if the degree of the polynomial  $d^{(k)}(\lambda, x_1, \dots, x_k)$  is positive; the set  $\Phi_0 \cup \dots \cup \Phi_{k-2}$  is nonempty if and only if the degrees of all polynomials which stand in the left part of System (17) are positive or the left parts of the equations of System (15) turn into the zero at the zero values of the arguments.

The described process of the construction of the set  $\Phi$  leads to a chain of sets of the equations and the systems of the equations. If the process does not break at some intermediate step, then this chain could be present as:

$$\begin{array}{c}
 f_j^{(m)}(\lambda, x_1, \dots, x_m) = 0 \quad (j = 1, \dots, n) \\
 \swarrow \quad \searrow \\
 d^{(m)}(\lambda, x_1, \dots, x_m) = 0 \quad f_j^{(m-1)}(\lambda, x_1, \dots, x_{m-1}) = 0 \quad (j = 1, \dots, n_{m-1}) \\
 \swarrow \quad \searrow \\
 d^{(m-1)}(\lambda, x_1, \dots, x_{m-1}) = 0 \quad f_j^{(m-2)}(\lambda, x_1, \dots, x_{m-2}) = 0 \quad (j = 1, \dots, n_{m-2}) \\
 \dots \dots \dots \\
 \swarrow \quad \searrow \\
 d^{(k+1)}(\lambda, x_1, \dots, x_{k+1}) = 0 \quad f_j^{(k)}(\lambda, x_1, \dots, x_k) = 0 \quad (j = 1, \dots, n_k) \\
 \swarrow \quad \searrow \\
 d^{(k)}(\lambda, x_1, \dots, x_k) = 0 \quad f_j^{(k-1)}(\lambda, x_1, \dots, x_{k-1}) = 0 \quad (j = 1, \dots, n_{k-1}) \\
 \dots \dots \dots \\
 \swarrow \quad \searrow \\
 d^{(3)}(\lambda, x_1, x_2, x_3) = 0 \quad f_j^{(2)}(\lambda, x_1, x_2) = 0 \quad (j = 1, \dots, n_2) \\
 \swarrow \quad \searrow \\
 d^{(2)}(\lambda, x_1, x_2) = 0 \quad f_j^{(1)}(\lambda, x_1) = 0 \quad (j = 1, \dots, n_1) \\
 \swarrow \\
 d^{(1)}(\lambda, x_1) = 0.
 \end{array} \tag{18}$$

Let us write out from (18) the equations participating in the construction of the set  $\Phi_k$  ( $k = 0, \dots, m - 1$ ), and we arrange them in the form

$$\begin{array}{c}
 f_j^{(m)}(\lambda, x_1, \dots, x_m) = 0 \quad (j = 1, \dots, n) \\
 \nwarrow \\
 f_j^{(m-1)}(\lambda, x_1, \dots, x_{m-1}) = 0 \quad (j = 1, \dots, n_{m-1}) \\
 \nwarrow \\
 \dots \dots \dots \\
 \nwarrow \\
 f_j^{(k+1)}(\lambda, x_1, \dots, x_{k+1}) = 0 \quad (j = 1, \dots, n_{k+1}) \\
 \nearrow \\
 d^{(k+1)}(\lambda, x_1, \dots, x_{k+1}) = 0 \quad (k = 0, \dots, m - 1).
 \end{array} \tag{19}$$

The diagram (19) shows that the solutions of the equation

$$d^{(k+1)}(\lambda, x_1, \dots, x_{k+1}) = 0$$

define  $(k + 1)$ -th components

$$x_{k+1}(\lambda) = \psi_{k+1}(x_1(\lambda), \dots, x_k(\lambda))$$

required solutions by  $x_1(\lambda), \dots, x_k(\lambda)$ . Further we pass to the system of the equations

$$f_j^{(k+1)}(\lambda, x_1, \dots, x_{k+1}) = 0 \quad (j = 1, \dots, n_{k+1}). \quad (20)$$

The solutions of System (51) define  $(k + 2)$ -th components

$$x_{k+2}(\lambda) = x_{k+2}(x_1(\lambda), \dots, x_k(\lambda), x_{k+1}(\lambda))$$

required solutions. Moving «upwards» we pass to the following system of the equations and etc. Finally we «reach» the last system of the equations

$$f_j^{(m)}(\lambda, x_1, \dots, x_m) = 0 \quad (j = 1, \dots, n). \quad (21)$$

The solutions of the system (21) define the last components

$$x_m(\lambda) = x_m(x_1(\lambda), \dots, x_k(\lambda), x_{k+1}(\lambda), \dots, x_{m-1}(\lambda))$$

required solutions.

From (18) and (19) follows [4, 8]:

**Theorem 1** *In the complex case System (1) has the finite number of small solutions if and only if the degrees of the polynomials  $d^{(i)}(\lambda, x_1, \dots, x_i)$  ( $i = 2, \dots, m$ ) are equal to the zero. Thus, if the degree of the polynomial  $d^{(1)}(\lambda, x_1)$  is equal to the zero then System (1) has no small solutions. If this degree is positive, then System (1) has the finite number of the small solutions.*

The simple examples show that in a real case the analogue of the Theorem 1 is false.

### 3 Analysis of the Rückert–Lefschetz scheme

The described scheme of the research of implicit functions defined by System (1) allows to describe the general structure of the small solutions  $x = x(\lambda)$  of this system. There appears a natural question: Is it possible, using the Rückert–Lefschetz scheme, to construct implicit functions (i.e. elements of the set  $\Phi$ ) determined with System (1)? The matter is that the calculations with analytical functions are usually realized through their expansion at the series of their variables (such calculations are usually called *approximate*). Moreover, actually only the first coefficients are used in the

calculations. Then there appears a new question: Are the first coefficients in Taylor expansions of solutions to System (1) determined with the first coefficients in the Taylor expansions of the left parts of this system in Taylor series?

Recall that the calculations in the Rückert–Lefschetz scheme described above are really realized in the rings of analytical functions at the zero of the variables  $(\lambda, x_1, \dots, x_{m-1}, x_m)$ ,  $(\lambda, x_1, \dots, x_{m-1})$ ,  $\dots$ ,  $(\lambda, x_1)$ ,  $\lambda$ . However, dealing with concrete systems of equations we must operate only with a finite number of coefficients in corresponding Taylor expansions of solutions and left hand parts of the system under consideration, or — as accepted to speak in a considered problem — to within the members of the higher order. At the first sight it seems that the similar approximate calculations probably to carry out for the Rückert–Lefschetz scheme described above. However it not so. More precisely, in the process of such calculation for concrete systems one can meet situations when the first coefficients of the Taylor expansions of solutions are not determined with the first coefficients of the Taylor expansions of the left parts of the system under consideration.

More exactly: *if calculations due to the Rückert–Lefschetz scheme are realized in the framework of calculations with the final number of coefficients (not in the rings of analytical functions or corresponding formal power series!) then it, generally speaking, does not allow to define a number of these implicit functions and jets of these solutions.* Thus, the Rückert–Lefschetz scheme has different properties in the framework of calculations in the rings of analytical functions and in the framework of calculations with the jets of the left hand sides of equations of System (1) and jets of its small solutions.

Of course, in the simplest case  $m = n = 1$  we do not meet with this problem. The Newton diagram method states that the answer to these questions is positive for simple solutions (a solution  $x(\lambda)$  of the scalar equation  $f(\lambda, x) = 0$  is simple if  $f'_x(\lambda, x(\lambda))$  is not zero) and negative for not simple solutions.

Although we are interested in the cases when  $m = n$ , under the realization of the Rückert–Lefschetz scheme the systems arise with  $m \neq n$  (more precisely, with  $m < n$ ). Let us consider one of such cases:  $m = 1$ ,  $n = 2$ . System (1) in this case has a form

$$\begin{cases} f_1(\lambda, x_1) = 0, \\ f_2(\lambda, x_1) = 0. \end{cases} \quad (22)$$

The scheme described above in this case is led to the calculation of the resultant  $f_{12}$  of the left parts of System (22) and to analysis of the equation

$$f_{12}(\lambda) = 0. \quad (23)$$

The equations of System (22) have a common solution if and only if the resultant  $f_{12}(\lambda) = 0$ . However, this fact is determined only an infinite number of the correspondent coefficients of  $f_{12}(\lambda)$ , and the calculations of the latter is required the knowledge of an infinite number of coefficients of the left hand parts of System (22). Thus, the fact of the solvability of the simplest overdetermined system does not

depend on the first coefficients in Taylor expansions of the left hand parts of the system under consideration. It is evident, that the analogous statement holds for arbitrary overdetermined systems.

The following simple case when  $m = 2$ ,  $n = 1$ ; System (1) in this case has a form

$$f_1(\lambda, x_1, x_2) = 0. \quad (24)$$

and, without the loss of generality, one can assume that the left hand side  $f_1(\lambda, x_1, x_2)$  of this equation is a polynomial with respect to  $x_2$ . In spite of the simplicity of this equation the analysis of its solutions is connected with serious difficulties and requires the using of Singularity Theory. However, the application of the Rückert–Lefschetz scheme, in the main (for us) case  $m = n$ , leads only to determined systems (if  $m = n = 2$ ) or overdetermined systems (if  $m = n > 2$ ). Thus, we need not in the analysis of System (24) and also more complicated underdetermined systems.

Let us consider the case:  $m = n = 2$ . In this case System (1) has a form

$$\begin{cases} f_1(\lambda, x_1, x_2) = 0, \\ f_2(\lambda, x_1, x_2) = 0. \end{cases} \quad (25)$$

The scheme described above in this case is led to the calculation of the resultant  $f_{12}$  of the left parts of the considered system and to analysis of the equation

$$f_{12}(\lambda, x_1) = 0. \quad (26)$$

From the description of the Rückert–Lefschetz scheme it follows that the first coefficients in Taylor expansions of the left hand sides of System (25) determine the first coefficients in Taylor expansion of the left hand sides of System (26). Then, in generic cases, in order to analyze System (26) it is possible to apply the Newton diagram method. This allows to define, generally speaking, the first members of all solutions to System (26).

Further, substituting these approximate solutions  $x_1(\lambda)$  in the equations of System (25) we received a system of compatible equations for the definition of the second components  $x_2(\lambda)$  of the solutions of System (25):

$$\begin{cases} f_1(\lambda, x_1(\lambda), x_2) = 0, \\ f_2(\lambda, x_1(\lambda), x_2) = 0. \end{cases} \quad (27)$$

This system is similar to System (22) (with the unknown  $x_2$  instead of  $x_1$ ), however, now we know that this system is solvable. Applying the Newton diagram method to each equations of System (27) one can construct the jets of all solutions to each equations of System (27). If there exists the only common jet of solutions to equations of System (27) then this jet is a jet of a common solution to both equations of System (27). In all other cases we can only state that System (27) is solvable but can not determined jets of common solutions to System (27).



Let  $\mathfrak{M}$  is a class of finite systems of type (1) with analytical left parts. We say that some scheme (algorithm)  $\mathfrak{S}$  of investigation of solutions of systems from  $\mathfrak{M}$  is *effective* if this scheme allows to define jets of all solutions of a system from  $\mathfrak{S}$  using only a finite number of the first coefficients in the Taylor expansions of the left hand parts of the system under consideration. It is evident, that the Rückert–Lefschetz scheme is non effective in the class  $\mathfrak{M}$  of finite systems of type (1) if in this class there exists systems without the property of roughness. So, the Rückert–Lefschetz scheme can be effective the class  $\mathfrak{M}$  contains only rough systems. However, the above-stated arguments prove the following statement.

**Theorem 2** *The Rückert–Lefschetz scheme of the construction of small solutions of system of type (1) is not effective at  $m = n > 1$  for a class of rough systems.*

Let us remind (see for example [4]) that the Newton diagram method of investigation of one scalar equation  $f(\lambda, x) = 0$  with an analytical left part is effective.

#### 4 Refinement of the Rückert–Lefschetz scheme

Below we give some standard complements to the Rückert–Lefschetz scheme, although these complements lie outside of our main results.

Let us consider a case, when  $k = 0$ . We present a polynomial  $q^{(1)}(\lambda, x_1) = d^{(1)}(\lambda, x_1)$  in the form of the product of prime multipliers over the ring  $K[\lambda, x_1]$  of analytical at the zero functions. Let  $p^{(1)}(\lambda, x_1)$  is one of prime multipliers of the polynomial  $q^{(1)}(\lambda, x_1)$ . Then each solution  $x_1(\lambda)$  of an equation

$$p^{(1)}(\lambda, x_1) = 0 \quad (30)$$

is a first component of an element in the set  $\Phi_0$ .

To define the second components  $x_2(\lambda)$  of elements in the set  $\Phi_0$  whose first components are solutions of Equation (30) we consider a following system from (18)

$$f_j^{(2)}(\lambda, x_1, x_2) = 0 \quad (j = 1, \dots, n_2). \quad (31)$$

The left parts of the equations in this system are the polynomials of  $x_2$  with coefficients from the ring  $K[\lambda, x_1]$  of analytical in the zero functions turning into zero at zero.

According to the Weierstrass preparation theorem let us replace coefficients of polynomials standing in the left part of System (31) with their remainders from division of them on the prime polynomial  $p^{(1)}(\lambda, x_1)$ . As result of such replacement System (31) pass to a system

$$\widehat{f}_j^{(2)}(\lambda, x_1, x_2) = 0 \quad (j = 1, \dots, n_2), \quad (32)$$

where  $\widehat{f}_j^{(2)}(\lambda, x_1, x_2)$  ( $j = 1, \dots, n_2$ ) are the polynomial of  $x_2$  whose coefficients are polynomials of  $x_1$  and which degrees are less than degree of the polynomial  $p^{(1)}(\lambda, x_1)$ .

As  $p^{(1)}(\lambda, x_1)$  is a prime polynomial then System (32) is possible to be consider as a system of algebraic equations from the unknown  $x_2$  in the field  $K(\lambda, x_1)$  which was obtained from the field  $K(\lambda)$  by adding an algebraic element  $x_1$ , where  $x_1$  is a solution of Equation (30). Since a concept of the greatest common divisor is defined in the ring of polynomials over the field, System (32) is equivalent to one equation

$$q^{(2)}(\lambda, x_1, x_2) = 0, \quad (33)$$

where  $q^{(2)}(\lambda, x_1, x_2)$  is a greatest common divisor of the polynomials standing in the left part of System (32). Thus, to define the second component  $x_2(\lambda)$  of elements in set  $\Phi_0$  which first components  $x_1(\lambda)$  are solutions of Equation (30), it is enough to find solutions of the algebraic equation (33).

Let  $p^{(2)}(\lambda, x_1, x_2)$  is one of prime multipliers of the polynomial  $q^{(2)}(\lambda, x_1, x_2)$ . We find the third components  $x_3(\lambda)$  of elements in the set  $\Phi_0$  whose first components  $x_1(\lambda)$  are solutions of Equations (30) and second components are solutions of an equation

$$p^{(2)}(\lambda, x_1, x_2) = 0. \quad (34)$$

With that end in view we consider a following system from (18)

$$f_j^{(3)}(\lambda, x_1, x_2, x_3) = 0 \quad (j = 1, \dots, n_3). \quad (35)$$

The left parts of equations of System (35) are polynomials of  $x_3$  with coefficients from the ring  $K[\lambda, x_1, x_2]$  of analytical in zero functions turning into zero at zero. As System (35) is considered together with Equation (34) it can be simplified. In the first place, in accordance with the Weierstrass preparation theorem, each coefficient of the left parts of System (35) is possible to replace with the remainder of its division by the prime polynomial  $p^{(2)}(\lambda, x_1, x_2)$ . And secondly, coefficients of obtained polynomials are possible be divided on the prime polynomial  $p^{(1)}(\lambda, x_1)$  according to the Weierstrass preparation theorem and replaced with the remainders from these divisions. As result System (35) pass to a system

$$\widehat{f}_j^{(3)}(\lambda, x_1, x_2, x_3) = 0 \quad (j = 1, \dots, n_3), \quad (36)$$

where  $\widehat{f}_j^{(3)}(\lambda, x_1, x_2, x_3) = 0$  ( $j = 1, \dots, n_3$ ) are polynomials of  $x_3$  whose coefficients are polynomials of  $x_2$  and whose degrees are less degree of the polynomial  $p^{(2)}(\lambda, x_1, x_2)$  and coefficients of these polynomials are polynomials of  $x_1$  whose degrees are less degree of the polynomial  $p^{(1)}(\lambda, x_1)$ .

As  $p^{(2)}(\lambda, x_1, x_2)$  is a prime polynomial then System (36) is possible to be consider as a system of algebraic equations from the unknown  $x_3$  in the field  $K(\lambda, x_1, x_2)$  which was obtained from the field  $K(\lambda, x_1)$  by adding an algebraic element  $x_2$ , where





to one algebraic equation and the components of corresponding element in set  $\Phi_0$  are defined by the roots of this algebraic equation:

$$\begin{cases} \psi(\lambda, \eta) = 0, \\ x_j(\lambda) = c_{j1}(\lambda) + c_{j2}(\lambda)\eta(\lambda) + \dots + c_{js}(\lambda)\eta^{s-1}(\lambda) \quad (j = 1, \dots, m). \end{cases} \quad (40)$$

Let us remind that a field of fractions  $K(\lambda)$  can be presented as

$$K(\lambda) = \left\{ \lambda^\theta \sum_{i=0}^p v_i \lambda^i : v_i \in \mathbb{C}, v_0 \neq 0, p \in \mathbb{N}, \theta \in \mathbb{Z} \right\}. \quad (41)$$

The field of fractions  $K(\lambda)$  is not algebraically closed, however its algebraic closure  $K^*(\lambda)$  is easily described

$$K^*(\lambda) = \bigcup_{r=1}^{\infty} K(\lambda^{\frac{1}{r}}), \quad (42)$$

where  $K(\lambda^{\frac{1}{r}})$  is a field of fractions of the rings  $K[\lambda^{\frac{1}{r}}]$  of analytical functions.

From the aforesaid follows that each element in the set  $\Phi_0$  is an analytical function of the parameter  $\lambda$  or an analytical function of  $\lambda^{\frac{1}{r}}$ .

Let us notice also that each system (39) defines one or several elements in the set  $\Phi_0$ .

Let us pass to the cases when  $k > 0$ . The basic arguments which was spent with the analysis of set  $\Phi_0$  are saved at the analysis of sets  $\Phi_k$ .

Let us present the polynomial  $q^{(k+1)}(\lambda, x_1, \dots, x_k, x_{k+1}) = d^{(k+1)}(\lambda, x_1, \dots, x_k, x_{k+1})$  in the form of the product of the prime polynomials over the ring  $K[\lambda, x_1, \dots, x_k]$  of analytical functions at the zero.

Let  $p^{(k+1)}(\lambda, x_1, \dots, x_k, x_{k+1})$  is one of the prime multipliers of the polynomial  $q^{(k+1)}(\lambda, x_1, \dots, x_k, x_{k+1})$ .

Let us choose arbitrary  $x_1(\lambda), \dots, x_k(\lambda)$ . Then each solution  $x_{k+1}$  of an equation

$$p^{(k+1)}(\lambda, x_1, \dots, x_k, x_{k+1}) = 0 \quad (43)$$

is  $(k+1)$ -th component of the element in the set  $\Phi_k$ .

To determine  $(k+2)$ -th components of the elements in the set  $\Phi_k$  whose first  $k$  components are arbitrary and  $(k+1)$ -th components  $x_{k+1}(\lambda)$  are solutions of Equations (43) we consider a system

$$f_j^{(k+2)}(\lambda, x_1, \dots, x_k, x_{k+1}, x_{k+2}) = 0 \quad (j = 1, \dots, n_{k+2}). \quad (44)$$

Repeating the arguments which was spent above at the construction of second components of solutions in the set  $\Phi_0$ , we pass to an equivalent system

$$\widehat{f}_j^{(k+2)}(\lambda, x_1, \dots, x_k, x_{k+1}, x_{k+2}) = 0 \quad (j = 1, \dots, n_{k+2}), \quad (45)$$

where  $\widehat{f}_j^{(k+2)}(\lambda, x_1, \dots, x_k, x_{k+1}, x_{k+2}) = 0$  ( $j = 1, \dots, n_{k+2}$ ) are the polynomials of  $x_{k+2}$  which coefficients are the polynomials of  $x_{k+1}$  and degrees are less than degree of the polynomial  $p^{(k+1)}(\lambda, x_1, \dots, x_k, x_{k+1})$ ; then we pass to an equation

$$q^{(k+2)}(\lambda, x_1, x_2, \dots, x_k, x_{k+1}, x_{k+2}) = 0, \tag{46}$$

where  $q^{(2)}(\lambda, x_1, x_2)$  is a greatest common divisor of the polynomials standing in the left part of System (45). Thus, to determine  $(k + 2)$ -th components  $x_{k+2}(\lambda)$  of elements in the set  $\Phi_k$  which  $(k + 1)$ -th components are solutions of Equation (43), it is enough to find the solutions of the algebraic equation (46).

Continuing similarly, as well as in a case  $k = 0$  we show that each component  $x_j(\lambda)$  ( $j = k + 1, \dots, m$ ) of the elements in the set  $\Phi_k$  is defined by an algebraic equation

$$p^{(j)}(\lambda, x_1, \dots, x_k, x_{k+1}, \dots, x_{j-1}, x_j) = 0, \tag{47}$$

whose left part is a prime polynomial of the variable  $x_j$  with coefficients from the ring  $K[\lambda, x_1, \dots, x_{j-1}]$  of analytical functions at the zero turning into the zero at the zero. Moreover, in turn these coefficients are polynomials of the variable  $x_{j-1}$  with coefficients from the ring  $K[\lambda, x_1, \dots, x_{j-2}]$  of analytical functions at the zero turning into the zero at the zero, and in turn coefficients of these polynomials are polynomials of the variable  $x_{j-2}$  with coefficients from the ring  $K[\lambda, x_1, \dots, x_{j-3}]$  of analytical functions at the zero turning into the zero at the zero and etc.

Collecting equations (43), (47) ( $j = k + 2, \dots, m$ ), we get that each element in set  $\Phi_k$  is defined by a system

$$\left\{ \begin{array}{l} p^{(m)}(\lambda, x_1, \dots, x_k, x_{k+1}, x_{k+2}, \dots, x_m) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ p^{(k+2)}(\lambda, x_1, \dots, x_k, x_{k+1}, x_{k+2}) = 0, \\ p^{(k+1)}(\lambda, x_1, \dots, x_k, x_{k+1}) = 0, \end{array} \right. \tag{48}$$

where  $x_1, \dots, x_k$  are free parameters and each function  $p^{(j)}(\lambda, x_1, \dots, x_j)$  ( $j = k + 1, \dots, m$ ) is a prime polynomial of the variable  $x_j$  with coefficients from the rings  $K[\lambda, x_1, \dots, x_{j-1}]$ . In other words  $(k + 1)$ -th component of an element in set  $\Phi_k$  can be considered as an element from algebraic expansion of the field of fractions  $K(\lambda, x_1, \dots, x_k)$  of the ring  $K[\lambda, x_1, \dots, x_k]$  of analytical functions at the zero,  $(k + 2)$ -th component of this element can be considered as an element from algebraic expansions of the field of fractions  $K(\lambda, x_1, x_2, \dots, x_k, x_{k+1})$  of the rings  $K[\lambda, x_1, x_1, \dots, x_k, x_{k+1}]$  of analytical functions at the zero, ..., at last, the last component  $x_m(\lambda)$  of this element can be considered as an element from algebraic expansions of the field of fractions  $K(\lambda, x_1, \dots, x_{m-1})$  of the ring  $K[\lambda, x_1, \dots, x_{m-1}]$  of analytical functions at the zero.

As the case  $k = 0$  (see, for example, [6, 9]), there are complex numbers  $a_i$ , ( $i = 1, \dots, m$ ) such that: (i) the function  $\eta(\lambda) = \sum_{i=k+1}^m a_i \cdot x_i(\lambda)$  satisfy to some algebraic

equation  $\psi(\lambda, x_1, \dots, x_k, \eta) = 0$  with coefficients from the field  $K(\lambda, x_1, \dots, x_k)$  and (ii) each function  $x_j(\lambda)$  ( $j = k+1, \dots, m$ ) lies at the field  $K(\lambda, \eta)$ , i. e. has the form  $x_j(\lambda) = c_{j1}(\lambda, x_1, \dots, x_k) + c_{j2}(\lambda, x_1, \dots, x_k)\eta(\lambda) + \dots + c_{js}(\lambda, x_1, \dots, x_k)\eta^{s-1}(\lambda)$ , where  $c_{j\sigma}(\lambda, x_1, \dots, x_k)$  ( $j = k+1, \dots, m, \sigma = 1, \dots, s$ ) are function from  $K(\lambda, x_1, \dots, x_k)$  and  $s$  is a degree of the equation  $\psi(\lambda, x_1, \dots, x_k, \eta) = 0$ . Thus System (48) appears equivalent to one algebraic equation and the components of the elements in set  $\Phi_k$  are defined by the root of this algebraic equation:

$$\begin{cases} \psi(\lambda, x_1, \dots, x_k, \eta) = 0, \\ x_j(\lambda) = c_{j1}(\lambda, x_1, \dots, x_k) + c_{j2}(\lambda, x_1, \dots, x_k)\eta(\lambda) + \\ \dots + c_{js}(\lambda, x_1, \dots, x_k)\eta^{s-1}(\lambda), \quad (j = k+1, \dots, m). \end{cases} \quad (49)$$

Thus, each function  $c_{j\sigma}(\lambda, x_1, \dots, x_k)$  ( $j = k+1, \dots, m, \sigma = 1, \dots, s$ ) can be presented in the form of fraction which numerator and denominator are elements of the ring  $K[\lambda, x_1, \dots, x_k]$  of analytical functions at the zero. Therefore it fallow to choice  $x_1(\lambda), \dots, x_k(\lambda)$  as the corresponding denominator of the functions  $c_{j\sigma}(\lambda, x_1, \dots, x_k)$  not turning into the zero at the zero.

Let us notice that the analogues of formulas (41) and (42) do not exist in the case  $k > 0$ .

Again as well as in the case  $k = 0$  each System (49) determines one or several elements in set  $\Phi_k$ .

From the spent above arguments follows

**Theorem 3** *In the field of fractions  $K(\lambda, x_1, \dots, x_k)$  ( $k = 0, \dots, m-1$ ) of the ring  $K[\lambda, x_1, \dots, x_k]$  of analytical functions at the zero for each element in the set  $\Phi_k$  there is a prime equation  $\psi(\lambda, x_1, \dots, x_k, \eta) = 0$  depending on free parameters  $\lambda$  and  $x_1(\lambda), \dots, x_k(\lambda)$ , which roots is defined by the components  $x_i(\lambda)$  ( $i = k+1, \dots, m-1$ ) of this element. The components  $x_i(\lambda)$  ( $i = k+1, \dots, m-1$ ) of the element in the set  $\Phi_k$  depend on free parameters  $\lambda$  and  $x_1(\lambda), \dots, x_k(\lambda)$ . Thus, the components of elements in set  $\Phi_0$  are the solutions of systems which equations have form (40), and the components of elements in the sets  $\Phi_k$  ( $k = 1, \dots, m-1$ ) are solutions of systems of equations (49).*

It has been above shown that the Rückert–Lefschetz scheme is not effective scheme for construction of small solutions of System (1). One can see that the refinement of this scheme is also "non effective" for calculation with coefficients of expansions of left hand sides of the equations in System (1). However, below we show how to get the effective scheme of the construction of some small solutions of System (1) by modifying the Rückert–Lefschetz scheme.

## 5 Modified Rückert–Lefschetz scheme

Applying the suitable change of variables and the Weierstrass preparation theorem, we pass from System (1) to the consideration of the equivalent system of algebraic

(with respect to  $x_n$ ) equations:

$$\tilde{f}_j^{(n)}(\lambda, x_1, \dots, x_n) = 0 \quad (j = 1, \dots, n). \tag{50}$$

Let us assume thus that  $m = n$ .

The set  $\mathfrak{D}_n$  of the trees with  $n$  vertexes is required to us. Let us remind that a tree with  $n$  vertexes is a coherent graph without simple cycles or, that is equivalent, a coherent graph with  $n$  vertexes and  $n - 1$  edges. The set  $\mathfrak{D}_n$  is finite; the number of its elements is equal to  $n^{n-2}$ . A vertex of a tree is called multiple if it is end vertex of more than one edge. We denote by  $\mu(D_n)$  the set of all multiple vertexes of a tree  $D_n$ .

Let  $D_n$  is a tree from the set  $\mathfrak{D}_n$ . Let us identify the vertexes of this tree with the system of Equations (50) (i.e. we enumerate vertexes of  $D_n$  and associate to  $j$ -th vertex of the tree  $D_n$  ( $1 \leq p \leq n$ ) the  $j$ -th equation of the system (50)). Further, to each edge  $\{j_1, j_2\}$  of the tree  $D_n$  ( $j_1$  and  $j_2$  are numbers of the end vertexes of the edge  $\{j_1, j_2\}$ ) we associate the resultant of the left parts of  $j_1$ -th and  $j_2$ -th equations of System (50). As a result we get the system of  $(n - 1)$  equations with  $n - 1$  unknowns:

$$\begin{cases} f_1^{(n-1)}(\lambda, x_1, \dots, x_{n-1}) = 0, \\ \dots\dots\dots \\ f_{n-1}^{(n-1)}(\lambda, x_1, \dots, x_{n-1}) = 0, \end{cases} \tag{51}$$

where  $f_i^{(n-1)}(\lambda, x_1, \dots, x_{n-1})$  ( $i = 1, \dots, n - 1$ ) are the resultants corresponding to the edges of  $D$ . We will be to assume that the left parts of this system are not zero.

As it is known (see, for example, [4])  $\tilde{x}(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda))$  is a solution of System (51), if  $x(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda), \xi_n(\lambda))$  is a solution of System (1). The opposite statement is not true. However, in some cases, it is succeed to state that for a given solution  $\tilde{x}(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda))$  of System (51) there exists a unique solution  $x(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda), \xi_n(\lambda))$  of System (1).

Let  $\tilde{x}(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda))$  is a small solution of System (51). Let us consider a system

$$\begin{cases} f_1(\lambda, \xi_1(\lambda), \dots, \xi_{n-1}(\lambda), x_n) = 0, \\ \dots\dots\dots \\ f_n(\lambda, \xi_1(\lambda), \dots, \xi_{n-1}(\lambda), x_n) = 0, \end{cases} \tag{52}$$

which is received from System (50) by the replacement in this system the components of the solution  $\tilde{x}(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda))$ . We say that the solution  $\tilde{x}(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda))$   $D_n$ -regular, if System (52) has the unique common simple solution  $x_n = \xi_n(\lambda)$ . According to this definition  $D_n$ -regular solution  $\tilde{x}(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda))$  uniquely determine the solution  $x(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda), \xi_n(\lambda))$  of System (50).

At first sight it seems that the definition of  $D_n$ -regular solution of System (51) is senseless since this definition requires that the components of the solution  $\tilde{x}(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda))$  are the first components of the corresponding solution of System

(50), i.e. this definition require that the solutions of System (50) are defined by the System (51) that evidently in the generic case is incorrectly. However, if in the generic case it is impossible to construct all solutions of System (50) by the solutions of System (51), in some natural cases it is probably to establish that the chosen solution of System (51) is  $D_n$ -regular (certainly if it is that) with the help of effective calculations (i.e. the calculations using only a finite number of coefficients in the expansion of the left parts of System (50)) and to construct the missing component of the solution of System (50).

The simple statement in this direction is

**Lemma 2** *If  $\tilde{x}(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda))$  is a small solution of the systems (51) and if each equation of System (52) with  $j \in \mu(D_n)$  has the unique solution then the solution  $\tilde{x}(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda))$  of System (51) is  $D_n$ -regular.*

Let us result the proof of the given statement. Assume that  $\tilde{x}(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda))$  is a small solution of System (51) and each equation of System (52) with  $j \in \mu(D_n)$  has a unique small solution. In this case, if  $j_1$  and  $j_2$  is connected with an edge from  $D_n$  then the corresponding equations

$$\begin{aligned} f_{j_1}(\lambda, \xi_1(\lambda), \dots, \xi_{n-1}(\lambda), x_n) &= 0, \\ f_{j_2}(\lambda, \xi_1(\lambda), \dots, \xi_{n-1}(\lambda), x_n) &= 0 \end{aligned} \tag{53}$$

have a common (and unique) solution.

Let us consider equations of System (52)

$$\begin{aligned} f_{j_1}(\lambda, \xi_1(\lambda), \dots, \xi_{n-1}(\lambda), x_n) &= 0, \\ f_{j_2}(\lambda, \xi_1(\lambda), \dots, \xi_{n-1}(\lambda), x_n) &= 0, \\ \dots\dots\dots & \\ f_{j_k}(\lambda, \xi_1(\lambda), \dots, \xi_{n-1}(\lambda), x_n) &= 0, \end{aligned} \tag{54}$$

where  $j_1, j_2, \dots, j_k \in \mu(D)$ . Since  $\mu(D_n)$  is a coherent subgraph of  $D_n$ , these equations also have a common (and unique) solution  $x_n = \xi_n(\lambda)$ .

Now let us consider a pair of the equations of System (53)

$$\begin{aligned} f_{j_1}(\lambda, \xi_1(\lambda), \dots, \xi_{n-1}(\lambda), x_n) &= 0, \\ f_{j_2}(\lambda, \xi_1(\lambda), \dots, \xi_{n-1}(\lambda), x_n) &= 0, \end{aligned} \tag{55}$$

where  $j_1 \in \mu(D_n)$ ,  $j_2 \notin \mu(D_n)$ ,  $j_1$  and  $j_2$  are connected with an edge from  $D_n$ . The first equation in this pair has a unique solution  $\xi_n(\lambda)$ . Simultaneously, both equations have a common solution. So,  $\xi_n(\lambda)$  is a solution of the second equation of this system.

Thus, System (54) has a unique common solution  $\xi_n(\lambda)$  and, furthermore, System (52) has a common (and unique) solution  $\xi_n(\lambda)$ . Hence System (50) has a common solution  $\tilde{x}(\lambda) = (\xi_1(\lambda), \dots, \xi_{n-1}(\lambda), \xi_n(\lambda))$  and so this solution is  $D_n$ -regular.





It is obvious that a common number of such different chains equals  $\prod_{i=2}^n i^{i-2}$ . A solution  $x(\lambda) = (x_1(\lambda), \dots, x_n(\lambda))$  of System (1) is called an *effectively computable solution* if  $(x_1(\lambda), \dots, x_{k-1}(\lambda))$  are  $D_k$ -regular solutions ( $k = 1, \dots, n$ ).

It is obvious that an effectively computable solution is a simple solution since each component of this solution is a simple solution of the corresponding system. The scheme of the construction of effectively computable solutions of System (1) we name as *modified Rückert–Lefschetz scheme*.

It is necessary to notice that in difference from the Rückert–Lefschetz scheme the modified Rückert–Lefschetz scheme does not allow to get the full description of the solutions of System (1), however in some cases the modified Rückert–Lefschetz scheme allows to construct effectively computable solutions of System (1).

From the spent above arguments follows

**Theorem 4** *The modified Rückert–Lefschetz scheme is an effective scheme for the construction in set of solutions of System (1) if and only if this set consists of only effectively computable solutions.*

## 6 Case of real effectively computable solutions

Above we supposed that the parameter  $\lambda$  and the unknowns  $x_1, \dots, x_m$  are complex numbers; the coefficients of the expansion in series  $f_j(\lambda, x_1, \dots, x_m)$  ( $j = 1, \dots, n$ ) accept the complex values. Therefore effectively computable solutions of System (1) constructed by the modified Rückert–Lefschetz scheme generally are complex. However, at a lot of applications as a rule represents the case when the parameter  $\lambda$  and the unknowns  $x_1, \dots, x_m$  are real numbers; the coefficients of expansion  $f_j(\lambda, x_1, \dots, x_m)$  ( $j = 1, \dots, n$ ) accept real values. We show how in such case to determine which of effectively computable solutions of System (1) constructed by the modified Rückert–Lefschetz scheme are real.

Let us use Newton’s diagram method for the construction of component  $x_k(\lambda)$  ( $k = 1, \dots, n$ ) of an effectively computable solution  $x(\lambda) = (x_1(\lambda), \dots, x_n(\lambda))$  of System (1).

The Newton’s diagram method allows to construct the set of solutions of the scalar equation from the parameter  $\lambda$ . Thus each solution of this equation can be presented in the form of series converging in some neighborhood of the zero.

The components  $x_k(\lambda)$  ( $k = 1, \dots, n$ ) of effectively computable solutions  $x(\lambda) = (x_1(\lambda), \dots, x_n(\lambda))$  of System (1) are defined by the scalar equations. Thus each component  $x_k(\lambda)$  ( $k = 1, \dots, n$ ) is a simple solution of the corresponding scalar equation and can be represented in kind of converging series in some neighborhood of the zero

$$x_k(\lambda) = \gamma_0 \lambda^{\frac{\tau_0}{\tau}} + \gamma_1 \lambda^{\frac{\tau_1}{\tau}} + \dots + \gamma_l \lambda^{\frac{\tau_l}{\tau}} + o(\lambda^{\frac{\tau_l}{\tau}})$$

$$(\gamma_i \in \mathbb{C}, \gamma_0 \neq 0, \tau, \tau_i \in \mathbb{N}, i = 0, \dots, l).$$



Therefore, if all members of the expansion  $x_k(\lambda)$  were real until the defining number  $r_k$  then all subsequent members of expansion  $x_k(\lambda)$  will be also real.

It is necessary to notice that reality of the coefficient  $\gamma_l$  not means reality of member  $\gamma_l \lambda^{\frac{\tau_l}{\tau}}$  of expansion  $x_k(\lambda)$  ( $k = 1, \dots, n$ ) since at different values  $\lambda$  ( $\lambda \geq 0$  and  $\lambda < 0$ ) the conditions on reality of the member  $\gamma_l \lambda^{\frac{\tau_l}{\tau}}$  will be various.

Let us notice also that the members of the expansion of the components  $x_1(\lambda), \dots, x_n(\lambda)$  of the simple solution  $x(\lambda) = (x_1(\lambda), \dots, x_n(\lambda))$  of System (1) are defined by a finite number of coefficients in the expansion at series of the left parts of the equations of System (1).

From the spent arguments follows

**Theorem 5** *The modified Rückert–Lefschetz scheme allows to determine real effectively computable solutions of System (1). Thus an effectively computable solution  $x(\lambda) = (x_1(\lambda), \dots, x_n(\lambda))$  is real if and only if first  $r_k$  members of the expansion of each component  $x_k(\lambda)$  ( $k = 1, \dots, n$ ) are real.*

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