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Existence and multiplicity of homoclinic orbits of a second-order differential difference equation via variational methods

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INVITED PAPER

Abstract: This paper is concerned with the existence of homoclinic orbits of the second order differential difference equation

$$\ddot{q}(t) - K_q(t,q(t)) + f(t,q(t+\tau),q(t),q(t-\tau)) = h(t) \quad .$$

By using critical point theory and variational methods, a nontrivial homoclinic orbit is obtained as a limit of a certain sequence of periodic solutions of an appropriate equation. As a result, we generalize the results obtained by Smets and Willem. Also, by applying a Symmetric Mountain Pass Lemma, we obtain infinitely many homoclinic orbits of the above equation.

Keywords: Homoclinic solutions; Differential difference equation; critical point theory.

1 Introduction

The purpose of this paper is to study the existence and multiplicity of homoclinic solutions to the differential difference equation

$$\ddot{q}(t) - K_q(t,q(t)) + f(t,q(t+\tau),q(t),q(t-\tau)) = h(t) \quad , \tag{1}$$

where $\tau > 0$ is a constant, $t \in \mathbf{R}$, $q \in \mathbf{R}^n$, $f(t,u_1,u_2,u_3) \in C(\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^n,\mathbf{R}^n)$, $f(t,0,0,0)=0$, $\frac{\partial f(t,u_1,u_2,u_3)}{\partial t} \neq 0$ and is $\tau-$periodic in t.

This kind of equation is the so-called mixed functional differential equation. Such equations arise in various applications. For example, in optimal control problems with delays, the Euler equation corresponding to the action functional often involves both advanced and

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delayed terms, see Pontryagin, etc [11]. In 1989, Rustichini [15, 16] studied a mixed functional differential equation arising from a competitive economy. In 1997, Wu and Zou [23] analyzed asymptotic behavior, periodic boundary value problems and wave solutions to a special mixed FDEs. A physical justification of (1) was also discussed by Schulman [17, 18] in the field of time symmetric electrodynamics and in absorber theory in Wheeler and Feynman [22].

In [18], Schulman studied the following equations

$$\ddot{q}(t)-\omega^2 q(t)=\tfrac{1}{2}\mu_1 q(t-\tau)+\tfrac{1}{2}\mu_2 q(t+\tau_1)+h(t) \tag{2}$$

and

$$\ddot{q}(t)-\omega^2 q(t)=\tfrac{1}{2}\mu_1 q(t-\tau)+\tfrac{1}{2}\mu_1 q(t+\tau)+h(t), \tag{3}$$

where μ_1,μ_2,ω,τ and τ_1 are given constants, $\tau,\tau_1>0$ and $h(t)$ is a given function.

Remark 1.1. Obviously, Eq.(2) and Eq.(3) are special cases of Eq.(1).

With $h(t)\equiv 0$, we have from (1) that

$$\ddot{q}(t)-K_q(t,q(t))+f(t,q(t+\tau),q(t),q(t-\tau))=0,$$

which arises in the study of traveling waves of the discrete sine-Gordon equation [10]

$$\ddot{q}_k=V'(q_{k+1}-q_k)-V'(q_k-q_{k-1})-K\sin(q_k),\quad k\in Z, \tag{4}$$

with a constant $K>0$ under the condition $K_q(t,q(t))=K\sin(q(t))$ and n=1. Eq.(4) describes the evolution of an infinite chain of atoms with elastic nearest neighbor interaction and an on-site potential, according to Newton's law. In [10], by employing variational methods, Kreiner and Zimmer obtained the existence of traveling heteroclinic, homoclinic and periodic waves.

In 1997, Smets and Willem [20] considered an infinite lattice of particles with nearest neighbor interaction:

$$\ddot{q}_k=V'(q_{k+1}-q_k)-V'(q_k-q_{k-1}),\quad k\in Z. \tag{5}$$

Using a variant of the mountain pass theorem, they proved the existence of solitary waves with prescribed speed.

In [10, 20], a solitary wave is a solution of (4) and (5) of the form

$$q_k(t)=u(k-ct),\quad k\in Z.$$

Substituting in (4) and (5), they obtained the second order forward-backward differential-difference equations

$$c^2u''(t)=V'(u(t+1)-u(t))-V'(u(t)-u(t-1))-K\sin(u(t)) \tag{6}$$

and

$$c^2u''(t)=V'(u(t+1)-u(t))-V'(u(t)-u(t-1)) \tag{7}$$

respectively.

Remark 1.2. Eq.(6) and Eq.(7) are special cases of Eq.(1).

In recent years several authors studied homoclinic orbits for Hamiltonian systems via critical point theory. In this paper we will establish the existence of homoclinic orbits to (1) by using a variational approach. For the existence of homoclinic orbits for second order differential equations we refer the reader to [2, 3, 4, 6, 7, 9, 12, 14] and for first order we refer the reader to [1, 5, 8, 19, 21, 24, 25] and the references therein.

We suppose that f, K and h in (1) satisfy the following assumptions:

(H_1) f is odd, i.e. for any $x \in R^n, f(t,-x) = -f(t,x)$;

(H_2) there are constants $k_1, k_2 > 0$ such that for all $(t,q) \in \mathbf{R} \times \mathbf{R}^n$

$$k_1|q|^2 \le K(t,q) \le k_2|q|^2 \quad \text{and} \quad \tfrac{1}{2}(q, K_q(t,q)) \le K(t,q) \le (q, K_q(t,q)),$$

where $K \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$ is $\tau-$periodic in t; here and in the sequel, $(\cdot,\cdot) : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ denotes the standard inner product in $\mathbf{R}^n$ and $|\cdot|$ the induced norm;

(H_3) there exists a continuously differentiable $\tau-$periodic function $F(t,v_1,v_2) \in C(\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n, \mathbf{R})$ with respect to t, such that

$$F'_{v_1}(t,q(t),q(t-\tau)) + F'_{v_2}(t,q(t+\tau),q(t)) = f(t,q(t+\tau),q(t),q(t-\tau));$$

(H_4) there is a constant $\beta > 2$ such that for every $t \in \mathbf{R}$ and $(v_1,v_2) \in \mathbf{R}^n \times \mathbf{R}^n \backslash \{(0,0)\}$

$$0 < \beta F(t,v_1,v_2) \le (F'_{v_1}(t,v_1,v_2),v_1) + (F'_{v_2}(t,v_1,v_2),v_2)$$

and $F(t,v_1,v_2) = 0$ if and only if $v_1 = v_2 = 0$.

Set $M := \sup\{F(t,v_1,v_2) : t \in [0,\tau], v_1^2 + v_2^2 = 1\}$, $d_1 := \min\{1, 2k_1\}$, $d_2 := \max\{1, 2k_2\}$, $4M < d_1$ and suppose that:

(H_5) $h : \mathbf{R} \to \mathbf{R}^n$ is a continuous and bounded function and $(\int_{\mathbf{R}} |h(t)|^2 dt)^{\frac{1}{2}} \le \frac{\eta}{2\rho}$, where $0 < \eta < d_1 - 4M$ and ρ is a positive constant which will be defined in Proposition 3.1 later.

As usual, a solution q of (1) is said to be homoclinic (to 0) if $q(t) \to 0$ as $t \to \pm\infty$. In addition, if $q \not\equiv 0$ then q is called a nontrivial homoclinic solution.

This paper is largely motivated by the work of Rabinowitz [12] in which the existence of nontrivial homoclinic solutions for the second order Hamiltonian system

$$\ddot{q} + V_q(t,q) = 0$$

was proved.

For the sake of completeness, two lemmas will be stated here which will be used in the proof of our main results.

Lemma 1.1 (Mountain Pass lemma)[13] Let E be a real Banach space and $I \in C^1(E, \mathbf{R})$

satisfy the **PS** condition. If further $I(0)=0$, and
(C_1) there exists constants ρ, $\alpha>0$ such that

$$I|_{\partial B_\rho(0)} \geq \alpha$$

and
(C_2) there exists $e\in E\backslash\overline{B_\rho(0)}$ such that $I(e)\leq 0$.
Then I possesses a critical value $c\geq\alpha$ given by

$$c=\inf_{g\in\Gamma}\max{}_{s\in[0,1]}I(g(s)),$$

where

$$\Gamma=\{g\in C([0,1],E)|g(0)=0 \quad \text{and} \quad g(1)=e\}.$$

Lemma 1.2 (Symmetric Mountain Pass lemma)[13] Let E be an infinite dimensional Banach space and let $I\in C^1(E,\mathbf{R})$ be even, satisfy the **PS** condition and $I(0)=0$. If $E=V\oplus X$, where V is finite dimensional, and I satisfies
(C_3) there exist constants ρ, $\alpha>0$ such that

$$I|_{\partial B_\rho\cap X}\geq\alpha, \quad \text{and}$$

(C_4) for each finite dimensional subspace $\widetilde{E}\subset E$, there is a $\gamma=\gamma(\widetilde{E})>0$ such that $I\leq 0$ on $\widetilde{E}\backslash B_\gamma$.
Then I possesses an unbounded sequence of critical values.

The rest of this paper is organized as follows. In Section 2 we establish a variational structure for (1) with periodic boundary value condition and state our main results. In particular we will show that under the assumptions of our main result the existence of $2k\tau-$periodic solutions is equivalent to the existence of critical points of some variational functional defined on a suitable Hilbert space. Finally, our main results will be proved in Section 3.

2 Variational structure and main results

For each $k\in\mathbf{N}$, let $E_k:=W^{1,2}_{2k\tau}(\mathbf{R},\mathbf{R}^n)$ be the Hilbert space of $2k\tau-$periodic functions on $\mathbf{R}$ with values in $\mathbf{R}^n$ under the norm

$$\|q\|_{E_k}:=(\int_{-k\tau}^{k\tau}|\dot{q}(t)|^2+|q(t)|^2dt)^{\frac{1}{2}}.$$

Furthermore, let $L^\infty_{[-k\tau,k\tau]}(\mathbf{R},\mathbf{R}^n)$ denote the space of $2k\tau$-periodic essentially bounded (measurable) functions from $\mathbf{R}$ into $\mathbf{R}^n$ equipped with the norm

$$\|q\|_{L^\infty_{[-k\tau,k\tau]}}:=ess\sup\{|q(t)|:t\in[-k\tau,k\tau]\}.$$

As in [9], a homoclinic solution of (1) will be obtained as a limit, as $k \to \pm\infty$, of a certain sequence of functions $q_k \in E_k$. We consider a sequence of systems of functional differential equations

$$\ddot{q}(t) - K_q(t,q(t)) + f(t,q(t+\tau),q(t),q(t-\tau)) = h_k(t), \tag{8}$$

where for each $k \in \mathbf{N}$, $h_k : \mathbf{R} \to \mathbf{R}^n$ is a $2k\tau-$periodic extension of the restriction of h to the interval $[-k\tau, k\tau]$ and q_k, a $2k\tau-$periodic solution of (8), will be obtained via the Mountain Pass Theorem.

Let

$$\phi_k(q) = (\int_{-k\tau}^{k\tau}[|\dot{q}(t)|^2 + 2K(t,q(t))]dt)^{\frac{1}{2}}. \tag{9}$$

By (H_2), we have

$$d_1\|q\|_{E_k}^2 \le \phi_k^2(q) \le d_2\|q\|_{E_k}^2. \tag{10}$$

Let $I_k : E_k \to \mathbf{R}$ be defined by

$$\begin{aligned} I_k(q) &= \int_{-k\tau}^{k\tau}[\tfrac{1}{2}|\dot{q}(t)|^2 + K(t,q(t))]dt + \int_{-k\tau}^{k\tau}(h_k(t),q(t))dt - \int_{-k\tau}^{k\tau}F(t,q(t),q(t-\tau))dt \\ &= \tfrac{1}{2}\phi_k^2(q) + \int_{-k\tau}^{k\tau}(h_k(t),q(t))dt - \int_{-k\tau}^{k\tau}F(t,q(t),q(t-\tau))dt. \end{aligned} \tag{11}$$

Then $I_k \in C^1(E_k, \mathbf{R})$ and it is easy to check that for any $q, y \in E_k$,

$$\begin{aligned} I_k'(q)y &= \int_{-k\tau}^{k\tau}(\dot{q}(t),\dot{y}(t))dt + \int_{-k\tau}^{k\tau}(K_q(t,q(t)) + h_k(t), y(t))dt \\ &\quad - \int_{-k\tau}^{k\tau}(F_{v_1}'(t,q(t),q(t-\tau)),y(t))dt - \int_{-k\tau}^{k\tau}(F_{v_2}'(t,q(t),q(t-\tau)),y(t-\tau))dt. \end{aligned} \tag{12}$$

By the periodicity of $q(t)$ and $F(t,q(t),q(t-\tau))$ with respect to t, we get

$$\int_{-(k+1)\tau}^{(k-1)\tau}(F_{v_2}'(t+\tau,q(t+\tau),q(t)),y(t))dt = \int_{-k\tau}^{k\tau}(F_{v_2}'(t,q(t+\tau),q(t)),y(t))dt.$$

Thus

$$\begin{aligned} I_k'(q)y &= \int_{-k\tau}^{k\tau}(-\ddot{q}(t) + K_q(t,q(t) + h_k(t), y(t))dt \\ &\quad - \int_{-k\tau}^{k\tau}(F_{v_1}'(t,q(t),q(t-\tau)) + F_{v_2}'(t,q(t+\tau),q(t)),y(t))dt \end{aligned}$$

and

$$\begin{aligned} I_k'(q)q &\le \phi_k^2(q) + \int_{-k\tau}^{k\tau}(h_k(t),q(t))dt - \int_{-k\tau}^{k\tau}(F_{v_1}'(t,q(t),q(t-\tau)),q(t))dt \\ &\quad - \int_{-k\tau}^{k\tau}F_{v_2}'(t,q(t+\tau),q(t)),q(t))dt. \end{aligned} \tag{13}$$

Therefore, the corresponding Euler equation of functional I_k is the following equation

$$\ddot{q}(t) - K_q(t,q(t)) + F_{v_1}'(t,q(t),q(t-\tau)) + F_{v_2}'(t,q(t+\tau),q(t)) = h_k(t). \tag{14}$$

Moreover, under assumption (H_3), it is clear that critical points of I_k are classical $2k\tau-$periodic solutions of (8).

We now state our main results

Theorem 2.1. Under the assumptions $(H_2)-(H_5)$, the system (1) possesses a nontrivial homoclinic solution $q \in W^{1,2}(\mathbf{R},\mathbf{R}^n)$ such that $\dot{q}(t) \to 0$ as $t \to \pm\infty$.

Remark 2.1. Theorem 2.1 extends both Theorem 7 and Theorem 8 in [20].

If $h(t) \equiv 0$ and $f(t,\cdot,\cdot,\cdot)$ is an odd function for any $t \in \mathbf{R}$, we show that the system (1) has infinitely many homoclinic solutions.

Theorem 2.2. If $h(t) \equiv 0$ and the conditions $(H_1)-(H_4)$ are satisfied then the system (1) possesses infinitely many nontrivial homoclinic solutions $q \in W^{1,2}(\mathbf{R},\mathbf{R}^n)$ such that $\dot{q}(t) \to 0$ as $t \to \pm\infty$.

3 Proof of the main results

In order to prove Theorem 2.1 and Theorem 2.2, some propositions are needed. We begin with a result which is a direct consequence of estimations made by Rabinowitz in [12].

Proposition 3.1. There is a positive constant ρ such that for each $k \in \mathbf{N}$ and $q \in E_k$ the following inequality holds:

$$\|q_k\|_{L^\infty_{[0,2k\tau]}} \leq \rho \|q_k\|_{E_k}. \tag{15}$$

Proposition 3.2. For every $t \in [0,\tau]$ the following inequalities hold:

$$F(t,v_1,v_2) \leq F(t,\tfrac{v_1}{|v|},\tfrac{v_2}{|v|})|v|^\beta \quad \text{if} \quad 0<|v|\leq 1, \tag{16}$$

$$F(t,v_1,v_2) \geq F(t,\tfrac{v_1}{|v|},\tfrac{v_2}{|v|})|v|^\beta \quad \text{if} \quad |v|\geq 1. \tag{17}$$

Proof. Let $g:[0,+\infty) \longrightarrow [0,+\infty)$ be defined as follows:

$$g(\zeta) = F(t,\zeta^{-1}v_1,\zeta^{-1}v_2)\zeta^\beta.$$

From (H_4), we have

$$\begin{aligned} g'(\zeta) &= \zeta^{\beta-1}[-(F'_{v_1}(t,\zeta^{-1}v_1,\zeta^{-1}v_2),\zeta^{-1}v_1) \\ &\quad -(F'_{v_2}(t,\zeta^{-1}v_1,\zeta^{-1}v_2),\zeta^{-1}v_2)+\beta F(t,\zeta^{-1}v_1,\zeta^{-1}v_2)] \\ &\leq 0. \end{aligned}$$

This shows that the function $F(t,\zeta^{-1}v_1,\zeta^{-1}v_2)\zeta^\beta$ is nonincreasing. Hence (16) and (17) follow.

Proposition 3.3. Let $m := \inf\{F(t,v_1,v_2): t \in [0,\tau], v_1^2+v_2^2=1\}$, $\zeta \in \mathbf{R}\backslash\{0\}$ and $v_1,v_2 \in E_k\backslash\{0\}$. Then we have

$$\int_{-k\tau}^{k\tau} F(t,\zeta v_1,\zeta v_2)dt \geq m|\zeta|^\beta \int_{-k\tau}^{k\tau} |v(t)|^\beta dt - 2km\tau. \tag{18}$$

Proof. Fix $\zeta \in \mathbf{R}\backslash\{0\}$ and $v_1, v_2 \in E_k\backslash\{0\}$. Set

$$D_k = \{t \in [-k\tau, k\tau] : |\zeta v| \leq 1\}$$

and

$$\overline{D_k} = \{t \in [-k\tau, k\tau] : |\zeta v| > 1\}.$$

From (17), we get

$$\begin{aligned} \int_{-k\tau}^{k\tau} F(t, \zeta v_1, \zeta v_2)dt &\geq \int_{\overline{D_k}} F(t, \zeta v_1, \zeta v_2)dt \geq \int_{\overline{D_k}} F(t, \tfrac{\zeta v_1}{|\zeta v|}, \tfrac{\zeta v_2}{|\zeta v|})|\zeta v|^\beta dt \\ &\geq m \int_{\overline{D_k}} |\zeta v|^\beta dt \\ &\geq m \int_{-k\tau}^{k\tau} |\zeta v|^\beta dt - m \int_{D_k} |\zeta v|^\beta dt \\ &\geq m|\zeta|^\beta \int_{-k\tau}^{k\tau} |v(t)|^\beta dt - 2km\tau. \end{aligned}$$

Proposition 3.4. Let $Y : [0, +\infty) \longrightarrow [0, +\infty)$ be defined as follows : $Y(0) = 0$ and

$$Y(s) = \max_{t\in[0,\tau], 0<v_1^2+v_2^2\leq s} \frac{(F'_{v_1}(t, v_1, v_2), v_1) + (F'_{v_2}(t, v_1, v_2), v_2)}{v_1^2 + v_2^2}$$

for $s > 0$. Then Y is continuous, nondecreasing, $Y(s) > 0$ for $s > 0$ and $Y(s) \to +\infty$ as $s \to +\infty$.

It is easy to verify this fact applying (H_4) and Proposition 3.2.

For each $k \in \mathbf{N}$, let $L^2_{[-k\tau,k\tau]}$ denote the Hilbert space of $2k\tau-$periodic functions on $\mathbf{R}$ with values in $\mathbf{R}^n$ under the norm

$$\|q\|_{L^2_{[-k\tau,k\tau]}} = (\int_{-k\tau}^{k\tau} |q(t)|^2 dt)^{\frac{1}{2}}.$$

Let $h_k : \mathbf{R} \to \mathbf{R}^n$ be a $2k\tau-$periodic extension of $h_k|_{[-k\tau,k\tau]}$ onto $\mathbf{R}$. From (H_5) it follows that

$$\|h_k\|_{L^2_{[-k\tau,k\tau]}} \leq \frac{\eta}{2\rho}. \tag{19}$$

Lemma 3.5 If f, F, K and h satisfy $(H_2) - (H_5)$, then for every $k \in \mathbf{N}$ the system (8) possesses a $2k\tau-$periodic solution.

Proof. It is clear that $I_k(0) = 0$. We now show that I_k satisfies the Palais-Smale condition. Assume that $\{q_m\}_{m\in\mathbf{N}}$ in E_k is a sequence such that $\{I_k(q_m)\}_{m\in\mathbf{N}}$ is bounded and $I'_k(q_m) \to 0$ as $m \to +\infty$. Then there exists a constant $d_3 > 0$ such that

$$|I_k(q_m)| \leq d_3, \quad \|I'_k(q_m)\|_{E_k^*} \leq d_3 \quad \text{for every} \quad m \in \mathbf{N}, \tag{20}$$

where E_k^* denotes the dual space of E_k.

We first prove that $\{q_m\}_{m\in\mathbf{N}}$ is bounded.
Note from (H_4) and (11) we have

$$\begin{aligned}\phi_k^2(q_m) &\le \tfrac{2}{\beta}\textstyle\int_{-k\tau}^{k\tau}(F'_{v_1}(t,q_m(t),q_m(t-\tau))+F'_{v_2}(t,q_m(t+\tau),q_m(t)),q_m(t))dt\\ &+2I_k(q_m)-2\textstyle\int_{-k\tau}^{k\tau}(h_k(t),q_m(t))dt.\end{aligned} \tag{21}$$

From (13) and (21) we get

$$(1-\tfrac{2}{\beta})\phi_k^2(q_m)\le 2I_k(q_m)-\tfrac{2}{\beta}I'_k(q_m)q_m-(2-\tfrac{2}{\beta})\textstyle\int_{-k\tau}^{k\tau}(h_k(t),q_m(t))dt. \tag{22}$$

Also from (22) we have

$$(1-\tfrac{2}{\beta})\phi_k^2(q_m)\le 2I_k(q_m)+[(2-\tfrac{2}{\beta})\|h_k\|_{L^2_{[0,2k\tau]}}+\tfrac{2}{\beta}\|I'_k(q_m)\|_{E_k^*}]\|q_m\|_{E_k}. \tag{23}$$

Combining (10) with (19), (20) and (23) we get

$$(1-\tfrac{2}{\beta})d_1\|q_m\|^2_{E_k}-2d_3-[\tfrac{2}{\beta}d_3+(2-\tfrac{2}{\beta})\tfrac{\eta}{2\rho}]\|q_m\|_{E_k}\le 0. \tag{24}$$

Since $\beta>2$, (24) shows that $\{q_m\}_{m\in\mathbf{N}}$ is bounded in E_k. Hence we can extract a subsequence of $\{q_{n_k}\}_{n_k\in\mathbf{N}}$ such that $\{q_{n_k}\}_{n_k\in\mathbf{N}}$ converges to q in E_k(weakly). On the other hand, (15) and (24) imply $q_m\to q$ uniformly on $[-k\tau,k\tau]$. Hence

$$(I'_k(q_m)-I'_k(q))(q_m-q)\to 0$$

and

$$\|q_m-q\|_{L^2_{[0,2k\tau]}}\to 0.$$

Set

$$\begin{aligned}\Phi &= \textstyle\int_{-k\tau}^{k\tau}((K_q(t,q(t)-K_{q_m}(t,q_m(t)),q_m(t)-q(t))dt\\ &+\textstyle\int_{-k\tau}^{k\tau}(F'_{v_1}(t,q_m(t),q_m(t-\tau)),q_m(t)-q(t))dt\\ &+\textstyle\int_{-k\tau}^{k\tau}(F'_{v_2}(t,q_m(t+\tau),q_m(t)),q_m(t)-q(t))dt\\ &-\textstyle\int_{-k\tau}^{k\tau}(F'_{v_1}(t,q(t),q(t-\tau)),q_m(t)-q(t))dt\\ &-\textstyle\int_{-k\tau}^{k\tau}(F'_{v_2}(t,q(t+\tau),q(t)),q_m(t)-q(t))dt.\end{aligned}$$

It is easy to check that $\Phi\to 0$ as $m\to+\infty$. Moreover, an easy computation shows that

$$(I'_k(q_m)-I'_k(q))(q_m-q)=\|\dot{q}_m-\dot{q}\|_{L^2_{[0,2k\tau]}}-\Phi,$$

and so $\|\dot{q}_m-\dot{q}\|_{L^2_{[0,2k\tau]}}\to 0$. Consequently, $\|q_m-q\|_{E_k}\to 0$.

We now show that there exist constants ρ, $\alpha > 0$ independent of k such that every I_k satisfies assumption (C_1) of Lemma 1.1 with these constants. Assume that $\|q\|_{L^{\infty}_{[0,2k\tau]}} \leq \frac{\sqrt{2}}{2}$. From (16) we have

$$\begin{aligned} \int_{-k\tau}^{k\tau} F(t,q(t),q(t-\tau))dt &\leq \int_{-k\tau}^{k\tau} F\left(t,\tfrac{q(t)}{|u(t)|},\tfrac{q(t-\tau)}{|u(t)|}\right)|u(t)|^{\beta} dt \\ &\leq M \int_{-k\tau}^{k\tau} |u(t)|^{\beta} dt \\ &\leq M \int_{-k\tau}^{k\tau} |u(t)|^{2} dt \\ &\leq 2M\|q\|_{E_k}^2, \end{aligned} \tag{25}$$

where $u(t) = \sqrt{q^2(t) + q^2(t-\tau)}$.

Also from (9)-(11), (19) and (25) we obtain

$$\begin{aligned} I_k(q) \geq\ & \tfrac{1}{2} d_1 \|q\|_{E_k}^2 - 2M\|q\|_{E_k}^2 - \|h_k\|_{L^2_{[0,2k\tau]}} \|q\|_{L^2_{[0,2k\tau]}} \\ &\geq \tfrac{1}{2} d_1 \|q\|_{E_k}^2 - 2M\|q\|_{E_k}^2 - \tfrac{\eta}{2\rho}\|q\|_{E_k} \\ &\geq \tfrac{1}{2}(d_1 - \eta - 4M)\|q\|_{E_k}^2 + \tfrac{\eta}{2}\|q\|_{E_k}^2 - \tfrac{\eta}{2\rho}\|q\|_{E_k}. \end{aligned} \tag{26}$$

Note that (H_5) implies $(d_1 - \eta - 4M) > 0$. Set

$$\rho = \tfrac{1}{\rho}, \quad \alpha = \tfrac{d_1 - \eta - 4M}{2\rho^2}.$$

From (15), if $\|q\|_{E_k} = \rho$ then (26) gives

$$I_k(q) \geq \alpha.$$

It remains to prove that for every $k \in N$ there exists $e_k \in E_k$ such that $\|e_k\|_{E_k} > \rho$ and $I_k(e_k) \leq 0$. From (9)-(11), (18) and (19) we have that for every $\zeta \in \mathbf{R}\backslash\{0\}$ and $q \in E_k\backslash\{0\}$

$$\begin{aligned} I_k(\zeta q) \leq\ & \tfrac{1}{2} d_2 |\zeta|^2 \|q\|_{E_k}^2 + |\zeta| \tfrac{\eta}{2\rho} \|q\|_{L^2_{[0,2k\tau]}} \\ & + 2km\tau - m|\zeta|^{\beta} \int_{-k\tau}^{k\tau} |u(t)|^{\beta} dt. \end{aligned} \tag{27}$$

Take $Q \in E_1$ such that $Q(\pm\tau) = 0$. Since $\beta > 2$ and $m > 0$, (27) implies that there exists $\xi \in \mathbf{R}\backslash\{0\}$ such that $\|\xi Q\|_{E_1} > \rho$ and $I_1(\xi Q) < 0$.

Set

$$e_1(t) = \xi Q(t)$$

and

$$e_k(t) = \begin{cases} e_1(t) & \text{for} \quad |t| \leq \tau \\ 0 & \text{for} \quad \tau < |t| \leq k\tau, \end{cases} \tag{28}$$

for $k > 0$. Then $e_k \in E_k$, $\|e_k\|_{E_k} = \|e_1\|_{E_1} > \rho$ and $I_k(e_k) = I_1(e_1) < 0$ for every $k \in \mathbf{N}$. From Lemma 1.1, I_k possesses a critical value c_k given by

$$c_k = \inf_{g\in\Gamma_k} \max{}_{s\in[0,1]} I_k(g(s)), \tag{29}$$

where

$$\Gamma_k = \{g \in C([0,1], E_k) | g(0) = 0 \quad \text{and} \quad g(1) = e_k\}.$$

Hence, for every $k \in \mathbf{N}$, there is $q_k \in E_k$ such that

$$I_k(q_k) = c_k, \quad I_k'(q_k) = 0. \tag{30}$$

The function q_k is a desired classical $2k\tau-$periodic solution of (HS_k). Since $c_k > 0$, q_k is a nontrivial solution even if $h_k(t) = 0$.

Lemma 3.6. Let $\{q_k\}_{k\in\mathbf{N}}$ be the sequence given by (30). There exists a q_0 such that $q_k \to q_0$ in $C^1_{loc}(\mathbf{R}, \mathbf{R}^n)$ as $k \to +\infty$.

Proof. The first step in the proof is to show that the sequences $\{c_k\}_{k\in\mathbf{N}}$ and $\{\|q_k\|_{E_k}\}_{k\in\mathbf{N}}$ are bounded. For every $k \in \mathbf{N}$, let $g_k : [0,1] \to E_k$ be a curve given by

$$g_k(s) = se_k,$$

where e_k is determined by (28). Then $g_k \in \Gamma_k$ and $I_k(g_k(s)) = I_1(g_1(s))$ for all $k \in \mathbf{N}$ and $s \in [0,1]$. Therefore, by (29),

$$c_k = \max{}_{s\in[0,1]} I_1(g_1(s)) \equiv M_0$$

independently of $k \in \mathbf{N}$.
As $I_k'(q_k) = 0$, we have from (H_4), (11) and (12) that

$$\begin{aligned} c_k &= I_k(q_k) - \tfrac{1}{2} I_k'(q_k) q_k \\ &\geq (\tfrac{\beta}{2} - 1) \textstyle\int_{-k\tau}^{k\tau} F(t, q_k(t), q_k(t-\tau)) dt + \tfrac{1}{2} \textstyle\int_{-k\tau}^{k\tau} (h_k(t), q_k(t)) dt. \end{aligned} \tag{31}$$

From (31) we obtain

$$\textstyle\int_{-k\tau}^{k\tau} F(t, q_k(t), q_k(t-\tau)) dt \leq \frac{1}{\beta-2} [2c_k - \int_{-k\tau}^{k\tau} (h_k(t), q_k(t)) dt]. \tag{32}$$

Next from (9)-(11), (19) and (32) we have

$$\|q_k\|^2_{E_k} \leq M_1. \tag{33}$$

To see this note

$$\begin{aligned} d_1 \|q_k\|^2_{E_k} &\leq \phi_k^2(q_k) \\ &= 2c_k - 2 \textstyle\int_{-k\tau}^{k\tau} (h_k(t), q_k(t)) dt + 2 \int_{-k\tau}^{k\tau} F(t, q_k(t), q_k(t-\tau)) dt \\ &\leq 2M_0 + \tfrac{\eta}{\rho} \|q_k\|_{E_k} + \tfrac{2}{\beta-2} [2M_0 + \tfrac{\eta}{2\rho} \|q_k\|_{E_k}] \\ &= \tfrac{2\beta M_0}{\beta-2} + \tfrac{\eta(\beta-1)}{\rho(\beta-2)} \|q_k\|_{E_k}. \end{aligned} \tag{34}$$

Since $d_1 > 0$ and all coefficients of (34) are independent of k, we see that there is a $M_1 > 0$ independent of k such that (33) holds.
From (15) and (33) we have

$$\|q_k\|_{L^\infty_{[0,2k\tau]}} \leq \rho M_1 \equiv M_2 \quad \text{for every} \quad k \in \mathbf{N}. \tag{35}$$

Since q_k satisfies (14), we have if $t \in [-k\tau, k\tau]$

$$\begin{aligned} |\ddot{q}_k(t)| \quad &\leq |h_k(t)| + |K_{q_k}(t, q_k)| \\ &+ |F'_{v_1}(t, q(t), q(t-\tau))| + |F'_{v_2}(t, q(t+\tau), q(t))|. \end{aligned} \tag{36}$$

Therefore, (H_2), (H_3), (H_5), (35) and (36) imply that there is a $M_3 > 0$ independent of k such that

$$\|\ddot{q}_k\|_{L^\infty_{[0,2k\tau]}} \leq M_3. \tag{37}$$

From (35) and (37) we have

$$\begin{aligned} |\dot{q}_k(t)| \quad &= |\textstyle\int_{\gamma_k}^t \ddot{q}_k(s)ds + \dot{q}_k(\gamma_k)| \\ &\leq \textstyle\int_{t-1}^t |\ddot{q}_k(s)|ds + |q_k(t) - q_k(t-1)| \\ &\leq M_3 + 2M_2. \end{aligned}$$

Thus for every $k \in \mathbf{N}$

$$\|\dot{q}_k\|_{L^\infty_{[0,2k\tau]}} \leq M_4.$$

Lemma 3.7. The function q_0 determined by Lemma 3.6 is the desired homoclinic solution of (1).

To prove this we use a result which was established by M. Izydorek and J. Janczewska in [9] which we now state.

Proposition 3.8. Let $q : \mathbf{R} \to \mathbf{R}^n$ be a continuous mapping such that $\dot{q} \in L^2_{loc}(\mathbf{R}, \mathbf{R}^n)$. For every $t \in \mathbf{R}$ the following inequality holds:

$$|q(t)| \leq \sqrt{2}[\textstyle\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|q(s)|^2 + |\dot{q}(s)|^2)ds]^{\frac{1}{2}}. \tag{38}$$

Proof of Lemma 3.7. The proof will be divided into three steps.
Step 1: We prove that $q_0(t) \to 0$, as $t \to \pm\infty$.
Note we have

$$\begin{aligned} \textstyle\int_{-\infty}^{+\infty} (|q_0(t)|^2 + |\dot{q}_0(t)|^2)dt \quad &= \lim_{j \to +\infty} \textstyle\int_{-j\tau}^{j\tau} (|q_0(t)|^2 + |\dot{q}_0(t)|^2)dt \\ &= \lim_{j \to +\infty} \lim_{k \to +\infty} \textstyle\int_{-j\tau}^{j\tau} (|q_{n_k}(t)|^2 + |\dot{q}_{n_k}(t)|^2)dt. \end{aligned}$$

Clearly, by (34), for every $j \in \mathbf{N}$ there exists $n_j \in \mathbf{N}$ such that for all $k \geq n_j$ we have

$$\textstyle\int_{-j\tau}^{j\tau} (|q_{n_k}(t)|^2 + |\dot{q}_{n_k}(t)|^2)dt \leq \|q_{n_k}\|^2_{E_{n_k}} \leq M_1^2.$$

Letting $k \to +\infty$, we get

$$\int_{-j\tau}^{j\tau}(|q_0(t)|^2 + |\dot{q}_0(t)|^2)dt \le M_1^2,$$

and now, letting $j \to +\infty$, we have

$$\int_{-\infty}^{+\infty}(|q_0(t)|^2 + |\dot{q}_0(t)|^2)dt \le M_1^2,$$

and so

$$\int_{|t|\ge m}(|q_0(t)|^2 + |\dot{q}_0(t)|^2)dt \to 0, \quad \text{as} \quad m \to +\infty. \tag{39}$$

Then (39) shows that our claim holds.

Step 2: We now show that $\dot{q}_0(t) \to 0$, as $t \to \pm\infty$.

Note from (38) that

$$\begin{aligned} |\dot{q}_0(t)|^2 &\le 2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\dot{q}_0(s)|^2 ds + 2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{q}_0(s)|^2 ds \\ &\le 2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|q_0(s)|^2 + |\dot{q}_0(s)|^2)ds + 2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{q}_0(s)|^2 ds. \end{aligned}$$

Since we have (39) and (3.30) it suffices to prove that

$$\int_m^{m+1} |\ddot{q}_0(t)|^2 dt \to 0, \quad \text{as} \quad m \to +\infty. \tag{40}$$

By (1) we obtain

$$\begin{aligned} \int_m^{m+1} |\ddot{q}_0(t)|^2 dt &= \int_m^{m+1}(|K_q(t,q_0(t)) - f(t,q_0(t+\tau),q_0(t),q_0(t-\tau))|^2 dt \\ &+ \int_m^{m+1} |h(t)|^2)dt + 2\int_m^{m+1}(K_q(t,q_0(t)),h(t))dt \\ &-2\int_m^{m+1}(f(t,q_0(t+\tau),q_0(t),q_0(t-\tau)),h(t))dt. \end{aligned}$$

Since $f(t,0,0,0) = 0, K_q(t,0) = 0$ for all $t \in \mathbf{R}$, $q_0(t) \to 0$, as $t \to \pm\infty$ and $\int_m^{m+1} |h(t)|^2 dt \to 0$, as $m \to \pm\infty$, (40) follows.

Step 3: Finally we show that if $h \equiv 0$ then $q_0 \not\equiv 0$.

By the definition of Y, we obtain

$$\begin{aligned} 2Y(\|q_k\|_{L^\infty_{[0,2k\tau]}})\|q_k\|_{E_k}^2 &\ge \int_{-k\tau}^{k\tau}(F'_{v_1}(t,q_k(t),q_k(t-\tau)),q_k(t))dt \\ &+ \int_{-k\tau}^{k\tau}(F'_{v_2}(t,q_k(t+\tau),q_k(t)),q_k(t))dt \end{aligned} \tag{41}$$

for every $k \in \mathbf{N}$. Since $I'_k(q_k)q_k = 0$, (12) gives

$$\begin{aligned} \int_{-k\tau}^{k\tau}(F'_{v_1}(t,q_k(t), \quad & q_k(t-\tau)) + F'_{v_2}(t,q_k(t+\tau),q_k(t)),q_k(t))dt \\ &\ge \int_{-k\tau}^{k\tau} |\dot{q}_k(t)|^2 dt + \int_{-k\tau}^{k\tau}(K_q(t,q_k(t)),q_k(t))dt. \end{aligned} \tag{42}$$

From (H_2), (41) and ((42)) we have

$$Y(\|q_k\|_{L^\infty_{[0,2k\tau]}})\|q_k\|^2_{E_k} \geq \min\{\tfrac{1}{2},k_1\}\|q_k\|^2_{E_k},$$

and hence

$$Y(\|q_k\|_{L^\infty_{[0,2k\tau]}}) \geq \min\{\tfrac{1}{2},k_1\}.$$

Consequently the properties of Y imply there is a $\lambda > 0$ (independent of k) such that

$$\|q_k\|_{L^\infty_{[0,2k\tau]}} \geq \lambda. \tag{43}$$

Now to complete the proof, observe that by the $\tau-$periodicity of F and K, whenever $q_0(t)$ is a $2k\tau-$periodic solution of (1), so is $q_0(t+j\tau)$ for all $j \in \mathbf{Z}$. Hence by replacing $q_k(t)$ earlier if necessary by $q_k(t+j\tau)$ for some $j \in [-k,k] \cap \mathbf{Z}$ it can be assumed that the maximum of $q_k(t)$ occurs in $[-\tau,\tau]$. Therefore if $q_k(t) \to 0$ in C^2_{loc} along our subsequence,

$$\|q_k\|_{L^\infty_{[-k\tau,k\tau]}} = \lim\max{}_{t\in[-\tau,\tau]}|q_k(t)| \to 0,$$

which contradicts (43).

Proof of Theorem 2.1: The result follows from Lemma 3.7.

Proof of Theorem 2.2: The condition (H_1) implies I_k is even. We know $I_k \in C^1(E,R)$, $I(0) = 0$ and I satisfies the **PS** condition. In order to prove Theorem 2.2 by using the Symmetric Mountain Pass Lemma, we shall show (C_3) and (C_4). From the proof of Theorem 2.1, (C_1) is true, so (C_3) is also true.

We now prove (C_4). Let $\widetilde{E}_k \subset E_k$ be an finite dimensional subspace. By (H_4), there exist some constants $\alpha_1 > 0, \alpha_2 > 0$ such that

$$F(t,v_1,v_2) \geq \alpha_1(\sqrt{v_1^2+v_2^2})^\beta - \alpha_2, \quad v_1,v_2 \in \widetilde{E}_k. \tag{44}$$

Then for any $\varphi \in \widetilde{E}_k, \varphi \neq 0$ and $\lambda > 0$, we have by (10) and (44)

$$\begin{aligned} I_k(\lambda\varphi) &= \tfrac{\lambda^2}{2}\phi_k(\varphi) - \textstyle\int_{-k\tau}^{k\tau} F(t,\lambda\varphi(t),\lambda\varphi(t-\tau))dt \\ &\leq \tfrac{d_2\lambda^2}{2}\|\varphi\|_{E_k} - \alpha_1\lambda^\beta \textstyle\int_{-k\tau}^{k\tau}(\sqrt{\varphi^2(t)+\varphi^2(t-\tau)})^\beta dt + 2k\tau\alpha_2. \end{aligned} \tag{45}$$

From (45) we have

$$\lim_{\lambda\to\infty} I_k(\lambda\varphi) = -\infty. \tag{46}$$

Now (46) implies that there exists a large enough λ such that $I_k(\lambda\varphi) \leq 0$. An argument similar to that in Theorem 2.1 (together with Lemma 1.2) guarantees that I_k possesses infinitely many nontrivial homoclinic solutions $q \in W^{1,2}(\mathbf{R},\mathbf{R}^n)$ such that $\dot{q}(t) \to 0$ as $t \to \pm\infty$. Thus the proof of Theorem 2.2 is complete.

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