SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR
SER. A: APPL. MATH. INFORM. AND MECH. vol. 8, 1 (2016), 75-87.

# Fixed Point Results in $G_q$-Metric Spaces with $W$-distance

**M. Abbas, H. Huang, M. Sarwar, M. Shoaib**

**Abstract:** The aim of this paper is to introduce the concept of $G_q$-metric space. We establish the existence of fixed points of Jachymski type mappings in the framework of $G_q$-metric spaces equipped with $w$-distance. Our results generalize and extend various results in the existing literature. Several examples to support our results are also presented.

**Keywords:** Fixed point, Jachymski function, $w$-distance, $G_q$-metric space.

## 1 Introduction and Preliminaries

The notion of distance plays an important role in geometric properties of Banach spaces as well as in different decision processes made in optimization theory. Gahler [7, 8] introduced 2-metric space in an attempt to generalize the concept of usual metric space. Ha et al. [11] showed the fact that 2-metric is not a continuous mapping with respect to its variables. There is irrelevance with the results between in ordinary metric spaces and in 2-metric spaces. These observations motivated Dhage [4] to introduce the concept of $D$-metric space (see also, [3, 5, 6]). Mustaf and Sims [14] mentioned some flaws in Dhage's theory of $D$-metric spaces. In 2006, Mustafa and Sims [15] introduced the concept of $G$-metric space to overcome the drawbacks in the existing theories concerning the generalizations of ordinary metric spaces. Based on $G$-metric spaces, several authors have obtained fixed and common fixed point results for mappings satisfying certain contractive conditions (see for example, [1, 16] and references contained therein). Saadati et al. [16] introduced the notion of $\Omega$-distance on $G$-metric spaces and obtained fixed point results by utilizing this concept. Recently, Alegre *et al.[2]* obtained an interesting result for the mapping satisfying some generalized contractive conditions on complete quasi-metric spaces involving $w$-distances. In this paper, we introduce the concept of $G_q$-metric space which generalizes the one of $G$-metric space. We obtain some fixed point results of Jachymski type mappings in the setup

Manuscript received November 21, 2015; accepted February 29, 2016.

M. Abbas is with the Lahore University of Management Sciences, Lahore-54792. Pakistan; H. Huang is with the School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, China; M. Sarwar and M. Shoaib are with the University of Malakand, Chakdara Dir(L), Khyber PakhtunKhwa, Pakistan.

of such spaces involving $w$-distances. Our results extend and strengthen previous known comparable results in the existing literature.

In the sequel we denote $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}^*$ as the set of all positive real numbers, the set of natural numbers and the set of all positive integers, respectively.

**Definition 1** *Let $X$ be a nonempty set. Suppose that a mapping $G : X \times X \times X \to \mathbb{R}^+$ satisfies:*

1. $G(x,y,z) = 0$ *if* $x = y = z \in X$;
2. $0 < G(x,x,y)$ *for all* $x,y \in X$ *with* $x \neq y$;
3. $G(x,x,y) \leq G(x,y,z)$ *for all* $x,y,z \in X$ *with* $y \neq z$;
4. $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ *(symmetry in all three variables);*
5. $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ *for all* $x,y,z,a \in X$ *(rectangle inequality).*

*Then $G$ is called a G-metric on $X$ and $(X,G)$ is called a G-metric space.*

**Definition 2** *Let $X$ be a nonempty set. Suppose that a mapping $G_q : X \times X \times X \to \mathbb{R}^+$ satisfies:*

1. $G_q(x,y,z) = 0$ *implies that* $x = y = z$ *for all* $x,y,z \in X$;
2. $G_q(x,x,y) \leq G_q(x,y,z)$ *for all* $x,y,z \in X$;
3. $G_q(x,y,z) \leq G_q(x,a,a) + G_q(a,y,z)$ *for all* $x,y,z,a \in X$ *(rectangle inequality).*

*Then $G_q$ is called a $G_q$-metric or G-quasi-metric on $X$ and $(X,G_q)$ is called a $G_q$-metric space or G-quasi metric space.*

**Remark 1** *Every G-metric space is a $G_q$-metric space but the converse is not true.*

**Example 1** *If $X = [0,\infty)$, then $G_q : X \times X \times X \to \mathbb{R}^+$ defined by*

$$G_q(x,y,z) = \max\{|x-y|, |y-z|\}$$

*is a $G_q$-metric on $X$. However, $G_q$ is not a G-metric on $X$. Indeed, $G_q(1,-1,0) \neq G_q(1,0,-1)$.*

**Definition 3** *Let $(X,G_q)$ be a $G_q$-metric space and $\{x_n\}$ a sequence in $X$. Then*

*(a)* $\{x_n\}$ *is called $G_q$-convergent if there exists $x \in X$ such that $G_q(x_n,x_m,x) \to 0$ as $n,m \to \infty$;*

*(b)* $\{x_n\}$ *is called a* $G_q$*-Cauchy sequence if for any* $\varepsilon > 0$*, there is a positive integer* $n_0$ *such that* $G_q(x_n, x_m, x_l) < \varepsilon$ *for all* $n, m, l > n_0$*;*

*(c)* $(X, G_q)$ *is called* $G_q$*-complete if every* $G_q$*-Cauchy sequence in* $X$ *is* $G_q$*-convergent in* $X$.

Saadati et al. [16] introduced the notion of $\Omega$-distance on $G$-metric spaces. We similarly define $w$-distance on $G_q$-metric spaces as follows:

**Definition 4** *Let* $X$ *be a* $G_q$*-metric space. A function* $q : X \times X \times X \to \mathbb{R}^+$ *is said to be a* $w$*-distance on* $X$ *if for any* $x, y, z, a \in X$*, the following conditions hold:*

*(W1)* $q(x,y,z) \leq q(x,a,a) + q(a,y,z)$*;*

*(W2)* $q(x,y,\cdot), q(x,\cdot,y) : X \to \mathbb{R}^+$ *are lower semi-continuous;*

*(W3) for each* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $q(x,a,a) \leq \delta$ *and* $q(a,y,z) \leq \delta$ *imply that* $G_q(x,y,z) \leq \varepsilon$.

For definitions and examples of $w$-distances on $G$-metric space, we refer to [2] and references mentioned therein.

Consistent with Alegre *et al.* [2], the following definition will be needed in the sequel.

**Definition 5** *A mapping* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is said to be a Meir-Keeler function if* $\phi(0) = 0$*, and for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $t \in \mathbb{R}^+$*,*

$$\varepsilon \leq t < \varepsilon + \delta \Rightarrow \phi(t) < \varepsilon, \tag{1.1}$$

Clearly, if $\phi$ is a Meir-Keeler function, then $\phi(t) < t$ for all $t > 0$.

**Definition 6** *A mapping* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is said to be a Jachymski function if* $\phi(0) = 0$*, and for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\varepsilon < t < \varepsilon + \delta \quad \textit{implies} \quad \phi(t) \leq \varepsilon, \quad \textit{for all} \quad t \in \mathbb{R}^+.$$

Note that Meir-Keeler function definitely be a Jachymski function. However, the converse does not hold in general (see [2]).

**Example 2** *Let* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be defined by*

$$\phi(t) = \begin{cases} \frac{t}{3}, & \textit{if } t \in [2,3], \\ 0, & \textit{otherwise}. \end{cases}$$

*Obviously,* $\phi(0) = 0$*. For* $\varepsilon > 0$*, we consider the following cases:*

*(1) If $0 < \varepsilon < 2$, then $\delta = 2 - \varepsilon$ with $\varepsilon \leq t < \varepsilon + \delta = 2$ implies that $\phi(t) = 0 < \varepsilon$.*

*(2) If $\varepsilon = 2$, then $\delta = 1$ with $\varepsilon \leq t < \varepsilon + \delta = 3$ gives that $\phi(t) = \frac{t}{3} < 1 < \varepsilon$;*

*(3) If $2 < \varepsilon < 3$, then choose $\delta = 3 - \varepsilon$. From $\varepsilon \leq t < \varepsilon + \delta = 3$, it follows that $\phi(t) = \frac{t}{3} < 1 < \varepsilon$;*

*(4) When $\varepsilon \geq 3$. Take $\delta$ such that $\varepsilon \leq t < \varepsilon + \delta$. As $\phi(3) = 1$ and $\phi(t) = 0$ for $t > 3$, so $\phi(t) < \varepsilon$.*

*Thus $\phi$ is a Meir-Keeler mapping and hence is a Jachymski function, too.*

Alegre *et al.* [2] proved the following result in the setting of quasi-metric spaces.

**Theorem 1** *Let $T$ be a self-mapping on a complete quasi-metric space $X$. Suppose that there exist a w-distance $q$ on $X$ and a Jachymski function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$. If for any $x, y \in X$, the following condition holds:*

$$q(Tx, Ty) \leq \phi(q(x, y)).$$

*Then $T$ has a unique fixed point $z \in X$ and $q(z, z) = 0$.*

## 2 Main result

We use the following lemma to prove the main result of this paper.

**Lemma 1** *Let $X$ be a $G_q$-metric space and $q$ a w-distance on $X$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in $X$, $\{\alpha_n\}$ and $\{\beta_n\}$ are null sequences in $[0, \infty)$ and $x, y, z, a \in X$. Then we have the following:*

1. *If $q(y, x_n, x_n) \leq \alpha_n$ and $q(x_n, y, z) \leq \beta_n$ for $n \in \mathbb{N}$. Then $G_q(y, y, z) < \varepsilon$.*
2. *If $q(y_n, x_n, x_n) \leq \alpha_n$ and $q(x_n, y_m, z) \leq \beta_n$ for $m > n$. Then $G_q(y_n, y_m, z) \to 0$.*
3. *If $q(x_n, x_m, x_l) \leq \alpha_n$ for $l, m, n \in \mathbb{N}$ with $n \leq m \leq l$. Then $\{x_n\}$ is a $G_q$-Cauchy sequence.*
4. *If $q(x_n, a, a) \leq \alpha_n$ for $n \in \mathbb{N}$. Then $\{x_n\}$ is a $G_q$-Cauchy sequence.*

**Proof** *Let $\varepsilon > 0$ be given. Choose $n_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta$ and $\beta_n \leq \delta$ for $n > n_0$. Then for $m > n \geq n_0$, we have $q(y_n, x_n, x_n) \leq \alpha_n < \delta$ and $q(x_n, y_m, z) \leq \beta_n < \delta$. Since $q$ is a w-distance on $X$, by (W3) it follows that $G_q(y_n, y_m, z) \to 0$ and hence $y_n \to z$. Thus (1) is completed, subsequently, (2) follows immediately from (1).*

*To prove (3), let $l > m > n \geq n_0$. Then*

$$q(x_n, x_{n+1}, x_{n+1}) \leq \alpha_n < \delta$$

*and*

$$q(x_{n+1}, x_m, x_l) \leq \alpha_{n+1} < \delta.$$

*Using (W3), we get* $G_q(x_n, x_m, x_l) \leq \varepsilon$ *and* $\{x_n\}$ *is a* $G_q$*-Cauchy sequence. Whereas, (4) is a special case of (3).*

■

Now we shall prove the main results of this paper as follows.

**Theorem 2** *Let* $X$ *be a complete* $G_q$*-metric space, and* $T$ *a self-mapping on* $X$*. Suppose that there exist a w-distance* $q$ *on* $X$ *and a Jachymski function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\phi(t) < t$*, for all* $t > 0$*. If for all* $x, y \in X$*, the following condition holds:*

$$q(Tx, Ty, Tz) \leq \phi(q(x, y, z)). \tag{2.1}$$

*Then* $T$ *has a unique fixed point* $z \in X$*. Moreover,* $q(z, z, z) = 0$*.*

**Proof** Let $x_0$ be a given point in $X$. Define a sequence $\{x_n\}$ in $X$ by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. By (2.1), it follows that

$$q(x_{n+1}, x_{n+2}, x_{n+2}) = q(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq \phi(q(x_n, x_{n+1}, x_{n+1})). \tag{2.2}$$

Put $r_n = q(x_n, x_{n+1}, x_{n+1})$ for all $n \in \mathbb{N}$. If there is $n_0 \in \mathbb{N}$ such that $r_{n_0} = 0$, then by repeated application of (2.2), we have $r_n = 0$ for all $n \geq n_0$. In particular,

$$r_{n_0} = q(x_{n_0}, x_{n_0+1}, x_{n_0+1}) = 0 \tag{2.3}$$

gives that

$$q(x_{n_0+1}, x_m, x_l) = 0 \ \text{ for } l = m \geq n_0 + 2. \tag{2.4}$$

As $q$ is a $w$-distance on $X$, so from (2.3) and (2.4) it follows that

$$G_q(x_{n_0}, x_m, x_l) = 0 \text{ for } l = m \geq n_0 + 2.$$

Thus $x_n = x_{n_0}$ for $n \geq n_0 + 2$. In this case, $x_{n_0}$ is a fixed point of $T$.

Assume that $r_n > 0$. Making using of (2.2), we arrive at

$$r_{n+1} \leq \phi(r_n) < r_n \text{ for } \ n \in \mathbb{N}. \tag{2.5}$$

Thus $\{r_n\}$ converges to some $r \in \mathbb{R}^+$. Obviously, $r_n > r$ for all $\in \mathbb{N}$. We prove that $r = 0$. Indeed, if $r > 0$, then there exists $\delta = \delta(r)$ such that $r < t < r + \delta$. By the given assumption, we have $\phi(t) \leq r$. Take $n_0 \in \mathbb{N}$ such that $r < r_n < r + \delta$, for any $n \geq n_0$, hence $\phi(r_n) \leq r$.

Also, from (2.2), it establishes that $r_{n+1} \leq r$ for all $n > n_0$. This a contradiction with (2.5). Accordingly, $r = 0$.

Now we show that $\{x_n\}$ is a $G_q$-Cauchy sequence. Let $\varepsilon > 0$. Take $\delta = \delta(\varepsilon) \in (0, \varepsilon)$ such that conditions (W3) and (1.1) hold. Also, for $\frac{\delta}{2} > 0$, choose $\delta_1 = \delta_1(\frac{\delta}{2}) \in (0, \frac{\delta}{2})$ such that (W3) and (1.1) hold. As $r_n \to 0$, there exists $k_0 \in \mathbb{N}$ such that $r_n < \delta_1$ for all $n \geq k_0$. By induction, we shall prove that for each $k \geq k_0$ and $n \in \mathbb{N}$, the following inequality (2.6) holds:

$$q(x_k, x_{n+k}, x_{n+k}) < \frac{\delta}{2} + \delta_1. \tag{2.6}$$

For any fixed $k \geq k_0$, we have

$$q(x_k, x_{1+k}, x_{1+k}) < \delta_1. \tag{2.7}$$

Hence (2.6) holds for $n = 1$. Assume that (2.6) holds for some $n \in \mathbb{N}$. We consider the following two cases:

(i) If

$$q(x_k, x_{n+k}, x_{n+k}) > \frac{\delta}{2}, \tag{2.8}$$

then by (1.1), we have

$$\phi(q(x_k, x_{n+k}, x_{n+k})) \leq \frac{\delta}{2}. \tag{2.9}$$

From (2.2) we have

$$q(x_{k+1}, x_{n+k+1}, x_{n+k+1}) \leq \frac{\delta}{2}. \tag{2.10}$$

By condition (W1), (2.7) and (2.10), we have

$$q(x_k, x_{n+k+1}, x_{n+k+1}) \leq q(x_k, x_{k+1}, x_{k+1}) + q(x_{k+1}, x_{n+k+1}, x_{n+k+1}) < \frac{\delta}{2} + \delta_1.$$

(ii) If

$$q(x_k, x_{n+k}, x_{n+k}) \leq \frac{\delta}{2}. \tag{2.11}$$

Then there are two possibilities: $q(x_k, x_{n+k}, x_{n+k}) = 0$ or $q(x_k, x_{n+k}, x_{n+k}) > 0$.

If $q(x_k, x_{n+k}, x_{n+k}) = 0$. Using (2.1), we have

$$q(x_{k+1}, x_{n+k+1}, x_{n+k+1}) = 0. \tag{2.12}$$

From (W1), (2.7) and (2.12), we get

$$\begin{aligned} q(x_k, x_{n+k+1}, x_{n+k+1}) &\leq q(x_k, x_{k+1}, x_{k+1}) + q(x_{k+1}, x_{n+k+1}, x_{n+k+1}) \\ &\leq \delta_1 + 0 < \frac{\delta}{2} + \delta_1. \end{aligned}$$

If $q(x_k, x_{n+k}, x_{n+k}) > 0$. Then we acquire that

$$\phi(q(x_k, x_{n+k}, x_{n+k})) < q(x_k, x_{n+k}, x_{n+k}) \leq \frac{\delta}{2}. \tag{2.13}$$

Using (W1), (2.2), (2.8) and (2.13), we have

$$\begin{aligned} q(x_k, x_{n+k+1}, x_{n+k+1}) &\leq q(x_k, x_{k+1}, x_{k+1}) + q(x_{k+1}, x_{n+k+1}, x_{n+k+1}) \\ &\leq q(x_k, x_{k+1}, x_{k+1}) + \phi(q(x_k, x_{n+k}, x_{n+k})) \\ &< \frac{\delta}{2} + \delta_1. \end{aligned}$$

Thus (2.6) is proved.

If we take $i \in \mathbb{N}$ with $i > k$. Then $i = n + k$ for some $n \in \mathbb{N}$. As $\delta_1 \in (0, \frac{\delta}{2})$, from (2.6) we have

$$q(x_k, x_i, x_i) = q(x_k, x_{n+k}, x_{n+k}) < \frac{\delta}{2} + \delta_1 < \delta.$$

As a result, by Lemma 2.1, $\{x_n\}$ is a $G_q$-Cauchy sequence in $X$. As $X$ is $G_q$-complete, there exists an element $z \in X$ such that $\lim_{n\to\infty} G_q(x_n, z, z) = 0$. Next, we shall show that $\lim_{n\to\infty} q(x_n, z, z) = 0$. Actually, for each $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that

$$q(x_k, x_{n+k}, x_{n+k}) < \varepsilon, \forall k \geq k_0, n \in \mathbb{N}.$$

Using the lower semi-continuity of $w$-distance $q$, we have

$$q(x_k, z, z) \leq \liminf_{n\to\infty} q(x_k, x_{n+k}, x_{n+k}) < \varepsilon.$$

Thus $q(x_k, z, z) < \varepsilon$ for all $k \geq k_0$ and hence the claim is satisfied. From (2.1) it follows that $\lim_{n\to\infty} q(x_{n+1}, Tz, Tz) = 0$. Using (W3), we have $\lim_{n\to\infty} G_q(x_n, Tz, Tz) = 0$. Consequently, $z = Tz$.

Note that $q(z, z, z) = 0$. If not, then

$$q(z, z, z) = q(Tz, Tz, Tz) \leq \phi(q(z, z, z)) < q(z, z, z)$$

gives a contradiction. Hence $q(z, z, z) = 0$. To prove the uniqueness of fixed point of $T$, let $u$ be another fixed point of $T$ such that $u \neq z$. If $q(u, z, z) > 0$, then we have

$$q(u, z, z) = q(Tu, Tz, Tz) \leq \phi(q(u, z, z)) < q(u, z, z),$$

a contradiction. Hence $q(u, z, z) = 0$. Similarly $q(u, u, u) = 0$. Using (W3), we have $G_q(u, z, z) = 0$. Thus $u = z$.

■

Now we present some corollaries of the above Theorem.

**Corollary 1** *Let $X$ be a complete $G_q$-metric space and $T$ a self-mapping on $X$. Suppose that there exist a w-distance $q$ on $X$ and a Jachymski function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$. If for all $x,y,z \in X$, the following condition holds:*

$$\int_0^{q(Tx,Ty,Tz)} \lambda(s)ds \leq \int_0^{\phi(q(x,y,z))} \lambda(s)ds, \tag{2.14}$$

*where $\lambda : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable, non-negative mapping on every compact subset of $[0,\infty)$ such that $\int_0^{\varepsilon} \lambda(s)ds > 0$ for each $\varepsilon > 0$. Then $T$ has a unique fixed point $z \in X$.*

**Proof** Define a mapping $\Psi : [0,\infty) \to [0,\infty)$ by $\Psi(t) = \int_0^t \lambda(s)ds$. Using (2.14) we obtain that

$$\Psi(q(Tx,Ty,Tz)) \leq \Psi(\phi(q(x,y,z)))$$

which implies that $q(Tx,Ty,Tz) \leq \phi(q(x,y,z))$. Thus by Theorem 2, $T$ has a unique fixed point.

■

**Corollary 2** *Let $X$ be a complete $G_q$-metric space, and $T$ a self-mapping on $X$. Suppose that there exist a w-distance $q$ on $X$ and a Meir-Keeler function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$. If for all $x,y,z \in X$, the following condition holds:*

$$q(Tx,Ty,Tz) \leq \phi(q(x,y,z)).$$

*Then $T$ has a unique fixed point $z \in X$.*

**Proof** Since every Meir-Keeler function is Jachymski function, so Corollary 2.2 follows from Theorem 2.1.

■

**Corollary 3** *Let $X$ be a complete $G_q$-metric space, and $T$ a self-mapping on $X$ . Suppose that there exist a w-distance $q$ on $X$ and a non-decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(0) = 0$, and $\phi^n(r) \to 0$ $(n \to \infty)$, $\forall\ r > 0$ . If for all $x,y,z \in X$, the following condition holds:*

$$q(Tx,Ty,Tz) \leq \phi(q(x,y,z)).$$

*Then $T$ has a unique fixed point $z \in X$.*

**Proof** To prove this, it is enough to show that $\phi$ is Meir-Keeler function. Assume on the contrary that $\phi$ is not a Meir-Keeler function. There exists $\varepsilon > 0$ and a sequence $\{r_n\}$ of positive real numbers such that

$$\varepsilon \leq r_n < \varepsilon + \delta \text{ and } \phi(r_n) \geq \varepsilon, \forall\, n \in \mathbb{N}.$$

Since $\phi$ is non-decreasing, then $\phi(r_n) \geq \varepsilon$. Hence $\phi^m(r_n) \geq \varepsilon$ $(\forall\, m \in \mathbb{N})$, a contradiction with the fact that $\phi^m(r) \to 0$ $(m \to \infty)$ for all $r > 0$. Therefore, $\phi$ is a Meir-Keeler and hence the result follows from Corollary 2.

■

The following example shows that the condition $\phi(t) < t$ $(\forall\, t > 0)$ from Theorem 2.1 can not be omitted.

**Example 3** *Let $X = \{0, 2\}$ and $G_q : X \times X \times X \to \mathbb{R}^+$ be a mapping defined by*

$$G_q(x,y,z) = |x-y| + |x-z|.$$

*Then $(X, G_q)$ is a complete $G_q$-metric space. Define the mappings $q : X \times X \times X \to \mathbb{R}^+$, $T : X \to X$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by*

$$q(x,y,z) = |x-y| + |x-z|,$$

$$Tx = \begin{cases} 2, & \text{if } x = 0, \\ 0, & \text{if } x = 2. \end{cases} \quad \phi(t) = \begin{cases} 2, & \text{if } t = 2, \\ 0, & \text{if } t = 4, \\ 0, & \text{otherwise.} \end{cases}$$

*For a given $\varepsilon > 0$, we consider the following cases:*
*(1) If $0 < \varepsilon < 2$, then $\delta = 2 - \varepsilon$ with $\varepsilon \leq t < \varepsilon + \delta = 2$ implies $\phi(t) = 0 < \varepsilon$;*
*(2) If $\varepsilon = 2$, take $\delta = \frac{1}{3}$, then from $\varepsilon < t < \varepsilon + \delta = \frac{7}{3}$ we have $\phi(t) = 0 < \varepsilon$;*
*(3) If $2 < \varepsilon < 4$, then $\delta = 4 - \varepsilon$ with $\varepsilon \leq t < \varepsilon + \delta = 4$ gives that $\phi(t) = 0 < \varepsilon$;*
*(4) If $\varepsilon = 4$, choose $\delta = \frac{1}{3}$, then from $\varepsilon < t < \varepsilon + \delta = \frac{13}{3}$ it follows that $\phi(t) = 0 < \varepsilon$;*
*(5) If $\varepsilon \geq 4$, then for any fixed $\delta$ with $\varepsilon \leq t < \varepsilon + \delta$ implies that $\phi(t) \leq \varepsilon$.*
*Thus $\phi$ is a Jachymski function. Consider the following table:*

*This table shows that condition (2.1) in Theorem 2 is satisfied but $T$ has no fixed point.*

**Example 4** *Let $X = \{0, \frac{1}{2}\}$ and $G_q$ be a mapping defined by*

$$G_q(x,y,z) = \max\{|x-y|, |y-z|\}.$$

*Then $X$ is a complete $G_q$-metric space. Define the mappings $q : X \times X \times X \to \mathbb{R}^+$, $T : X \to X$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $Tx = \frac{1}{2}$, $q(x,y,z) = \max\{|x-y|, |y-z|\}$, and*

$$\phi(t) = \begin{cases} \frac{1}{4}, & \text{if } t = \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

| $x$ | $y$ | $z$ | $Tx$ | $Ty$ | $Tz$ | $q(x,y,z)$ | $q(Tx,Ty,Tz)$ | $\phi(q(x,y,z))$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 2 | 2 | 2 | 0 | 0 | 0 |
| 0 | 0 | 2 | 2 | 2 | 0 | 2 | 2 | 2 |
| 0 | 2 | 0 | 2 | 0 | 2 | 2 | 2 | 2 |
| 0 | 2 | 2 | 2 | 0 | 0 | 4 | 4 | 4 |
| 2 | 0 | 0 | 0 | 2 | 2 | 4 | 4 | 4 |
| 2 | 0 | 2 | 0 | 2 | 0 | 2 | 2 | 2 |
| 2 | 2 | 0 | 0 | 0 | 2 | 2 | 2 | 2 |
| 2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |

*Clearly $\phi(0) = 0$. For $\varepsilon > 0$, consider the following cases:*
*(1) If $0 < \varepsilon < \frac{1}{2}$, then $\delta = \frac{1}{2} - \varepsilon$ with $\varepsilon \leq t < \varepsilon + \delta = \frac{1}{2}$ implies that $\phi(t) = 0 < \varepsilon$;*
*(2) If $\varepsilon = \frac{1}{2}$, then $\delta = \frac{1}{3}$. From $\varepsilon < t < \varepsilon + \delta = \frac{5}{6}$, it follows that $\phi(t) = 0 < \varepsilon$;*
*(3) If $\varepsilon \geq \frac{1}{2}$, then for any fixed $\delta$ with $\varepsilon \leq t < \varepsilon + \delta$ gives $\phi(t) < \varepsilon$.*
*So $\phi$ is a Jachymski function. From the following table, we are easy to see all the conditions of Theorem 2.1 are satisfied, hence $T$ has a unique fixed point $x = \frac{1}{2}$.*

| $x$ | $y$ | $z$ | $Tx$ | $Ty$ | $Tz$ | $q(x,y,z)$ | $q(Tx,Ty,Tz)$ | $\phi(q(x,y,z))$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 |
| 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{4}$ |
| 0 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{4}$ |
| 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{4}$ |
| $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{4}$ |
| $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{4}$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{4}$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 |

**Example 5** *Let $X = \{0,1,2\}$ and let the mappings $G_q : X \times X \times X \to \mathbb{R}^+$, $q : X \times X \times X \to \mathbb{R}^+$ be defined by*

$$q(x,y,z) = G_q(x,y,z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{|x|,|y|\} + \max\{|x|,|z|\}, & \text{otherwise.} \end{cases}$$

*Then $(X, G_q)$ is a complete $G_q$-metric space. Let the mappings $T : X \to X$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$*

*be defined by*

$$T(x) = \begin{cases} 0, & if\, x = 1, \\ \frac{x}{2}, & if\, x = \{0,2\}. \end{cases} \quad \phi(t) = \begin{cases} 1, & if\, t = 2, \\ 2, & if\, t = 3, \\ 3, & if\, t = 4, \\ 0, & otherwise. \end{cases}$$

*Clearly* $\phi(0) = 0$. *For* $\varepsilon > 0$ *we consider the following cases:*
*(1) If* $0 < \varepsilon < 1$, *then* $\delta = 1 - \varepsilon$ *with* $\varepsilon \leq t < \varepsilon + \delta = 1$ *implies that* $\phi(t) = 0 < \varepsilon$;
*(2) If* $\varepsilon = 1$. *Choose* $\delta = \frac{1}{3}$, *then* $\varepsilon < t < \varepsilon + \delta = \frac{4}{3}$ *gives* $\phi(t) = 0 < \varepsilon$;
*(3) If* $1 < \varepsilon < 2$, *then* $\delta = 2 - \varepsilon$ *with* $\varepsilon \leq t < \varepsilon + \delta = 2$ *implies that* $\phi(t) = 0 < \varepsilon$;
*(4) If* $\varepsilon = 2$, *take* $\delta = \frac{1}{3}$. *From* $\varepsilon < t < \varepsilon + \delta = \frac{7}{3}$ *it follow that* $\phi(t) = 0 < \varepsilon$;
*(5) If* $2 < \varepsilon < 3$, *then* $\delta = 3 - \varepsilon$ *with* $\varepsilon \leq t < \varepsilon + \delta = 3$ *gives that* $\phi(t) = 0 < \varepsilon$;
*(6) If* $\varepsilon = 3$, $\delta = \frac{1}{3}$ *with* $\varepsilon < t < \varepsilon + \delta = \frac{10}{3}$, *then* $\phi(t) = 0 < \varepsilon$;
*(7) If* $\varepsilon \geq 3$, *then for any fixed* $\delta$ *with* $\varepsilon \leq t < \varepsilon + \delta$ *implies that* $\phi(t) \leq \varepsilon$.

| $x$ | $y$ | $z$ | $Tx$ | $Ty$ | $Tz$ | $q(x,y,z)$ | $q(Tx,Ty,Tz)$ | $\phi(q(x,y,z))$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 2 | 0 | 0 | 1 | 2 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 | 2 | 0 | 1 |
| 0 | 1 | 2 | 0 | 0 | 1 | 3 | 1 | 2 |
| 0 | 2 | 0 | 0 | 1 | 0 | 2 | 1 | 1 |
| 0 | 2 | 1 | 0 | 1 | 0 | 3 | 1 | 2 |
| 0 | 2 | 2 | 0 | 1 | 1 | 4 | 2 | 3 |
| 1 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 1 |
| 1 | 0 | 2 | 0 | 0 | 1 | 3 | 1 | 2 |
| 1 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 1 |
| 1 | 1 | 1 | 0 | 0 | 0 | 2 | 0 | 1 |
| 1 | 1 | 2 | 0 | 0 | 1 | 3 | 1 | 2 |
| 1 | 2 | 0 | 0 | 1 | 0 | 3 | 1 | 2 |
| 1 | 2 | 1 | 0 | 1 | 0 | 3 | 1 | 2 |
| 1 | 2 | 2 | 0 | 1 | 1 | 4 | 2 | 3 |
| 2 | 0 | 0 | 1 | 0 | 0 | 4 | 2 | 3 |
| 2 | 0 | 1 | 1 | 0 | 0 | 4 | 2 | 3 |
| 2 | 0 | 2 | 1 | 0 | 1 | 4 | 2 | 3 |
| 2 | 1 | 0 | 1 | 0 | 0 | 4 | 2 | 3 |
| 2 | 1 | 1 | 1 | 0 | 0 | 4 | 2 | 3 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| *2* | *1* | *2* | *1* | *0* | *1* | *4* | *2* | *3* |
| *2* | *2* | *0* | *1* | *1* | *0* | *4* | *2* | *3* |
| *2* | *2* | *1* | *1* | *1* | *0* | *4* | *2* | *3* |
| *2* | *2* | *2* | *1* | *1* | *1* | *4* | *2* | *3* |

As a consequence, $\phi$ is a Jachymski function. From the above table, we are easy to see that all the conditions of Theorem 2 are satisfied and therefore $x = 0$ is the unique fixed point of $T$.

## Acknowledgement

The second author is thankful to the science and technology research project of the education department in Hubei Province of China (B2015137).

## References

[1] M. ABBAS, T. NAZIR, P. VETRO, *Common fixed point results for three maps in G-metric spaces*,*Filomat*, **25** 1-17, (2011).

[2] C. ALEGRE, J. MARN AND S. ROMAGUERA,*A fixed point theorem in generalized contraction involving w-distances on complete quasi-metric spaces*, Fixed Theory Appl., **40** (2014) 1-8.

[3] B. C. DHAGE,*Generalized metric space and mapping with fixed point*, Bull. Calcutta Math. Soc. **84** (1992), 329-336.

[4] B.C. DHAGE, *Generalized metric space and topological structure I* , An. Stiint. Univ., Al. I. Cuza Iasi, Mat, **46** 3-24,(2000).

[5] B. C. DHAGE, *On generalized metric spaces and topological structure II*, Pure Appl. Math. Sci., **40**(1994),37-41.

[6] B. C. DHAGE, *On continuity of mappings in D-metric spaces*, Bull. Calcutta Math. Soc **86** (1994), 503-508.

[7] S. GHLER, *Zur geometric 2-metrische rume. Rev. Roum. Math*, Pures Appl. XL (1966), 664-669.

[8] S. GHLER, *2-Metrische Rume und ihr topologische struktur*, Math. Nachr. **26** (1963), 115-148.

[9] L. GHOLIZADEH,*A fixed point theorem in generalized ordered metrice spaces with application*, J. Nonlinear Sci. Appl.,**6**(2013), 244-251.

[10] L. GHOLIZADAH, R. SAADATI, *Contractive mapping in Genralized ordered metric space with application in integral equations*,Math. Prob. Engin., **2011** (2011), 1-14.

[11] HA, KS, CHO, YJ, WHITE, A, *Strictly convex and strictly 2-convex 2-normed spaces*, Math. Jpn,**33** (1988), 375-384.

[12] J. JACHYMSKI,*Equivalent conditions and the meir-keeler type theorems*,J. Math. Anal. Appl., **194**(1995), 293-303.

[13] J. MATKOWSKI, *Integrable solutions of functional equations*, Diss. Math, **127** (1975), 1-68.

[14] Z. MUSTAFA, B. SIMS,*Some remarks concerning D-metric spaces*, In Proc. Inter. Conferences Fixed Point Theorey Appl., cValencia (Spain), (2003),189-198.

[15] Z. MUSTAFA, B. SIMS, A new approach to generalized metric spaces, *J. Nonlinear Convex Anal.*, **7**(2006), 289-297.

[16] R. SAADATI, S.M. VAEZPOUR, P. VETRO, B.E. RHOADES, *Fixed point theorems in generalized partially ordered G-metric spaces*, Math. Comput. Model., **52** (2010), 797-801.